

5. The following is a frequency distribution table of a grouped data with variable X. What is the mean ($\bar{x}$) and the variance (s^2) of the data respectively? (1)

[Figure: A two-row bordered frequency table with five columns. Top row: first cell labelled 'X', then the class intervals 3-7, 8-12, 13-17, 18-22. Bottom row: first cell labelled 'Frequency', then the values 4, 6, 8, 2 under those intervals respectively.]

- A. ✓ $\bar{x} = 12, s^2 = 21$ B. $\bar{x} = 12, s^2 = 25$
C. $\bar{x} = 13, s^2 = 9$ D. $\bar{x} = 13, s^2 = 16$

Answer: A

With midpoints 5, 10, 15, 20 and frequencies 4, 6, 8, 2, the mean is $\bar{x} = 240/20 = 12$, and the variance is $s^2 = \sum f(x - \bar{x})^2/N = 420/20 = 21$.

6. The expenditure of 100 families is given below. The mode of the data is 23.5, what are the values of f_1 and f_2 (1)

[Figure: A two-row bordered table with six columns. Top row: first cell labelled 'Expenditure', then the classes 0-9, 10-19, 20-29, 30-39, 40-49. Bottom row: first cell labelled 'No. of Families', then the values 14, 23, f_1 , 21, f_2 under those classes respectively.]

- A. ✓ $f_1 = 27, f_2 = 15$ B. $f_1 = 15, f_2 = 27$
C. $f_1 = 25, f_2 = 17$ D. $f_1 = 17, f_2 = 25$

Answer: A

The mode is 23.5, which lies in 19.5 – 29.5. Using the mode formula $23.5 = 19.5 + \frac{f_1 - 23}{(f_1 - 23) + (f_1 - 21)} \times 10$ yields $f_1 = 27$, and since the total is 100, $f_2 = 100 - (14 + 23 + 27 + 21) = 15$.

7. The first group of 10 children has a mean weight of 15.6kg, the second group of another 10 children has a mean weight of 16kg, and the third group of children has a mean weight of 20kg. If the mean weight of all the children is 17kg, what is the total number of children in all of the three group? (1)

- A. ✓ 28 B. 29
C. 30 D. 32

Answer: A

Setting up the weighted mean equation $\frac{10(15.6) + 10(16) + k(20)}{10 + 10 + k} = 17$ gives $316 + 20k = 340 + 17k$, so $k = 8$, giving a total of $10 + 10 + 8 = 28$ children.

8. Which of the following is a valid argument? (1)

- A. $\neg p \Rightarrow \neg q, q \vdash \neg p$ B. ✓ $p \Rightarrow \neg q, p, r \Rightarrow q \vdash \neg r$
C. $\neg p \vee q, r \Rightarrow p, r \vdash \neg q$ D. $\neg p, p \vee q, r \Rightarrow q \vdash \neg r$

Answer: B

From p and $p \Rightarrow \neg q$, Modus Ponens yields $\neg q$. Then, combined with $r \Rightarrow q$, Modus Tollens validly concludes $\neg r$.

9. Consider the following open propositions: $P(x)$ = x is a prime number, $C(x)$ = x is a composite number, and $E(x)$ = x is an even number, which one of the following has a truth value of True in the set of positive integers? (1)

- A. $(\forall x)[P(x) \Rightarrow \neg E(x)]$ B. $(\forall x)[P(x) \Rightarrow \neg E(x)]$
C. ✓ $(\forall x)[P(x) \Rightarrow \neg E(x)]$ D. $(\forall x)[P(x) \Rightarrow \neg E(x)]$

Answer: C

10. Which of the following functions is a one to one correspondence? (1)

- A. $f : [0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = |x|$ B. $f : \mathbb{R} \rightarrow [0, \infty)$ defined by $f(x) = x^2$
C. $f : \mathbb{R} \rightarrow [0, \infty)$ defined by $f(x) = 3^x$ D. $\checkmark f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \log_2 x$

Answer: D

The logarithmic function $f(x) = \log_2 x$ with domain $(0, \infty)$ and codomain $\mathbb{R}$ is strictly increasing (hence one-to-one) and its range is all of $\mathbb{R}$ (hence onto), making it a one-to-one correspondence.

11. If $f(x) = \sqrt{x^3}$ and $(f \circ g)(x) = \sqrt[4]{x}$, then what is the value of $g(8)$? (1)

- A. $\sqrt[3]{2}$ B. 2
C. $\checkmark \sqrt{2}$ D. $2\sqrt{2}$

Answer: C

Given $f(g(x)) = (g(x))^{3/2} = x^{1/4}$, solving for $g(x)$ gives $g(x) = x^{1/6}$. Evaluating at $x = 8 = 2^3$ yields $g(8) = (2^3)^{1/6} = 2^{1/2} = \sqrt{2}$.

12. Which one of the following is the inverse of $f(x) = 8x^3 + 2$? (1)

- A. $f^{-1}(x) = \frac{1}{8x^3 + 2}$ B. $\checkmark f^{-1}(x) = \frac{1}{2}\sqrt[3]{x-2}$
C. $f^{-1}(x) = 8x^{-3} + 2$ D. $f^{-1}(x) = \frac{1}{8}\sqrt[3]{x-2}$

Answer: B

Setting $y = 8x^3 + 2$, we solve for x : $y - 2 = 8x^3 \implies x^3 = \frac{y-2}{8} \implies x = \frac{1}{2}\sqrt[3]{y-2}$, so the inverse function is $f^{-1}(x) = \frac{1}{2}\sqrt[3]{x-2}$.

13. What is the solution set of $\frac{1}{1 + \frac{1}{x}} - \frac{1}{1 - \frac{1}{x}} = \frac{x + \frac{1}{x}}{x - \frac{1}{x}}$? (1)

- A. $\checkmark \{ \}$ B. $\{-1\}$
C. $\{1\}$ D. $\{-1, 1\}$

Answer: A

Simplifying both sides yields $\frac{-2x}{x^2-1} = \frac{x^2+1}{x^2-1}$, which leads to $(x+1)^2 = 0$, giving $x = -1$. However, $x = -1$ makes denominators zero in the original equation, so it is extraneous and the solution set is empty, $\{ \}$.

14. Which one of the following is true about the graph of $\frac{x^3}{x^3(x-1)}$? (1)

- A. the vertical asymptotes of the graph are $x = 0$ and $x = 1$
B. a horizontal asymptote of the graph is $y = 1$
C. $\checkmark$ the graph intersects its horizontal asymptote at a point $(-1, 0)$
D. the graph intersects the vertical line $x = 1$ at a point $(1, 2)$

Answer: C

The horizontal asymptote of the rational function is $y = 0$ because the degree of the denominator exceeds that of the numerator; setting the function equal to the asymptote $y = 0$ gives the intersection point $(-1, 0)$.

15. What is the equation of a line that passes through point (a, a) in xy -plane if it is parallel to a line that passes through points (a, b) and (b, a) where $a \neq b$? (1)

- A. $y = x$ B. $y = -x$
C. $y = -x + 2a$ D. $y = 2x - a$

Answer: C

The slope of the line passing through (a, b) and (b, a) is $m = \frac{a-b}{b-a} = -1$. The parallel line passing through (a, a) has equation $y - a = -1(x - a)$, which simplifies to $y = -x + 2a$.

16. What are the values of the center (C) and radius (r) of a circle $x^2 + y^2 - 4x + 6y = 5$? (1)

- A. $C = (-2, 3), r = 3\sqrt{2}$ B. $C = (2, -3), r = 3\sqrt{2}$
C. $C = (2, -3), r = 2\sqrt{3}$ D. $C = (-2, 3), r = 2\sqrt{3}$

Answer: B

Completing the square gives $(x - 2)^2 + (y + 3)^2 = 5 + 4 + 9 = 18$. Thus, the center is $C = (2, -3)$ and the radius is $r = \sqrt{18} = 3\sqrt{2}$.

17. What is the radius of the largest possible circle that can be inscribed in the ellipse given by $5(x - 1)^2 + 3y^2 = 15$? (1)

- A. $\sqrt{3}$ B. $\sqrt{5}$
C. 3 D. 5

Answer: A

In standard form, the ellipse is $\frac{(x-1)^2}{3} + \frac{y^2}{5} = 1$, with semi-minor axis $b = \sqrt{3}$ and semi-major axis $a = \sqrt{5}$. The largest circle that can be inscribed within the ellipse has a radius equal to the semi-minor axis, which is $\sqrt{3}$.

18. Suppose the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is reciprocal to that of the eccentricity of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through the focus of the ellipse, then what is the equation of the hyperbola? (1)

- A. $x^2 - 2y^2 = 2$ B. $x^2 - 3y^2 = 3$
C. $x^2/3 - y^2/2 = 1$ D. $x^2/2 - y^2/3 = 1$

Answer: B

The ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ has $c = \sqrt{3}$ and eccentricity $e = \frac{\sqrt{3}}{2}$. The hyperbola has vertex $a = \sqrt{3}$ and eccentricity $e_h = \frac{2}{\sqrt{3}}$, giving $c_h = 2$ and $b^2 = c_h^2 - a^2 = 1$, leading to the equation $x^2 - 3y^2 = 3$.

19. Let p and q stand for the statements "Nejat is intelligent" and "Almaze is hardworking". Respectively, which of the following represent the statement "Almaze is hardworking if Nejat is intelligent"? (1)

- A. $\neg p \wedge q$ B. $\neg p \vee q$
C. $p \wedge q$ D. $\neg q \vee p$

Answer: B

The conditional statement 'q if p' means 'if p, then q', symbolized as $p \implies q$. By conditional-disjunction equivalence, $p \implies q \equiv \neg p \vee q$.

20. Let f be differentiable function with $f(1) = -1$ and $f'(1) = 1$. If $g(x) = [f(2x + 1) + 2]^2$, then what is the value of $g'(0)$? (1)

A. ✓ 4
B. 2
C. -2
D. -4

Answer: A

By the chain rule, $g'(x) = 2[f(2x + 1) + 2] \cdot f'(2x + 1) \cdot 2 = 4[f(2x + 1) + 2]f'(2x + 1)$. Evaluating at $x = 0$ gives $g'(0) = 4[f(1) + 2]f'(1) = 4[-1 + 2](1) = 4$.

21. If $f(x) = \ln(x^2 + 2)$, then what is the value of $f''(1)$? (1)

A. 3/2
B. 5/9
C. 2/3
D. ✓ 2/9

Answer: D

The first derivative is $f'(x) = \frac{2x}{x^2+2}$. Differentiating again by the quotient rule gives $f''(x) = \frac{2(x^2+2) - 2x(2x)}{(x^2+2)^2} = \frac{4-2x^2}{(x^2+2)^2}$, so $f''(1) = \frac{4-2}{3^2} = \frac{2}{9}$.

22. If $x^2 + xy = 10$, then what is the value of $\frac{dy}{dx}$ when $x = 2$? (1)

A. ✓ -7/2
B. 2/7
C. 3/2
D. 7/2

Answer: A

When $x = 2$, $2^2 + 2y = 10 \implies y = 3$. Implicit differentiation yields $2x + y + x\frac{dy}{dx} = 0$, which at $(2, 3)$ gives $\frac{dy}{dx} = -\frac{2(2)+3}{2} = -\frac{7}{2}$.

23. What is the equation of the tangent line to the graph of $f(x) = 3x^2 + 4x - 5$ at $(1, 2)$? (1)

A. ✓ $10x - y - 8 = 0$
B. $-10x + y - 8 = 0$
C. $-10x - y - 8 = 0$
D. $10x + y - 8 = 0$

Answer: A

The slope of the tangent line is $f'(x) = 6x + 4$, so at $x = 1$, $m = 10$. The equation of the line is $y - 2 = 10(x - 1)$, which simplifies to $10x - y - 8 = 0$.

24. If $f(x) = \pi^2 + 1$, then what is the value of $f'(x)$? (1)

A. $2\pi + 1$
B. 2π
C. 2
D. ✓ 0

Answer: D

Since $\pi^2 + 1$ (or $2\pi + 1$) is a constant, the function $f(x)$ is a constant function whose derivative with respect to x is 0.

25. Suppose f is continuous on $[2, 6]$ and the only solutions of the equation $f(x) = 7$ are $x = 2$ and $x = 5$. If $f(3) = 9$, then one of the following Can Not be the value of $f(4)$? (1)

A. ✓ 5
B. 7.5
C. 8
D. 9

Answer: A

By the Intermediate Value Theorem, since f is continuous on $[2, 5]$ and $f(x) = 7$ only at $x = 2$ and $x = 5$, $f(x)$ cannot cross 7 on $(2, 5)$. Since $f(3) = 9 > 7$, $f(x) > 7$ for all $x \in (2, 5)$, so $f(4)$ cannot be 5.

26. What is the value of k so that $f(x) = \begin{cases} \frac{\tan 2x}{x}, & \text{if } x > 0 \\ k - e^x, & \text{if } x \leq 0 \end{cases}$ is continuous at $x = 0$? (1)
- A. 2
B. ✓ 3
C. 1
D. 0

Answer: B

For continuity at $x = 0$, we require $\lim_{x \rightarrow 0^+} f(x) = f(0)$. Here, $\lim_{x \rightarrow 0^+} \frac{\tan 2x}{x} = 2$ and $f(0) = k - e^0 = k - 1$, so $k - 1 = 2 \implies k = 3$.

27. If f is continuous at $x = 0$ and $g(x) = \sqrt{x} \left(2f(x) + \frac{3}{\sqrt{x}} \right)$ for all $x > 0$, then what is the value of $\lim_{x \rightarrow 0^+} g(x)$? (1)
- A. 0
B. 2
C. ✓ 3
D. 5

Answer: C

Expanding $g(x)$ gives $g(x) = 2\sqrt{x}f(x) + 3$. Taking the limit as $x \rightarrow 0^+$ yields $2(0)f(0) + 3 = 3$ since f is continuous at 0.

28. What is the value of $\lim_{x \rightarrow 0} \frac{\sin x \cos 2x}{x^2 + 3x}$? (1)
- A. ✓ $1/3$
B. $2/3$
C. 1
D. 2

Answer: A

We can rewrite the limit as $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \cdot \frac{\cos 2x}{x+3} = 1 \cdot \frac{1}{3} = \frac{1}{3}$.

29. If $a \neq 0$, then what is the value of $\lim_{x \rightarrow 0} \frac{\sin x \cos 2x}{x^2 + 3x}$? (1)
- A. $1/a^2$
B. ✓ $1/2a^2$
C. $1/6a^2$
D. 0

Answer: B

30. What are the values of λ and μ so that the system $\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ x + 2y + \lambda z = \mu \end{cases}$ has infinitely many solutions? (1)
- A. $\lambda \neq 3$ and $\mu \in \mathbb{R}$
B. $\lambda = 3$ and $\mu \neq 10$
C. ✓ $\lambda = 3$ and $\mu = 10$
D. $\lambda \in \mathbb{R}$ and $\mu = 10$

Answer: C

Subtracting the second equation from the third gives $(\lambda - 3)z = \mu - 10$. For the system to have infinitely many solutions, this must be an identity $0z = 0$, requiring $\lambda = 3$ and $\mu = 10$.

31. When $\begin{vmatrix} a & b & c \\ a & -a & a \\ a & a & -a \end{vmatrix} = a^3$, and $a \neq 0$, what is the solution of $\begin{bmatrix} a & b & c \\ a & -a & a \\ a & a & -a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$? (1)
- A. $x = 0, y = 2a, z = 2a$ B. $x = 1/a, y = -2a, z = 2a$
 C. $x = 1/a, y = 2/a, z = 2/a$ D. ✓ $x = 0, y = 2/a, z = 2/a$

Answer: D

Equations 2 and 3 give $x - y + z = 0$ and $x + y - z = 0$, which sum to $2x = 0 \implies x = 0$ and $y = z$. The determinant expands to $2a^2(b + c) = a^3$, so $b + c = a/2$; substituting $x = 0$ and $y = z$ into the first equation gives $(b + c)y = 1 \implies y = z = 2/a$.

32. If $z_1 = \frac{2-i}{1+i}$ and $z_2 = \frac{1+i}{1-i}$, then what is the value of $z_1 + 2z_2$? (1)
- A. $1 + i$ B. ✓ $\frac{i+1}{2}$
 C. $\frac{i+2}{2}$ D. $1 - i$

Answer: B

Simplifying gives $z_1 = \frac{(2-i)(1-i)}{2} = \frac{1-3i}{2}$ and $z_2 = \frac{(1+i)^2}{2} = i$. Thus, $z_1 + 2z_2 = \frac{1-3i}{2} + 2i = \frac{1+i}{2} = \frac{i+1}{2}$.

33. Which one of the following is the conjugate of $|3 + 4i| - \frac{25i}{3 + 4i}$? (1)
- A. $5 + 3i$ B. ✓ $1 + 3i$
 C. $3 - 5i$ D. $1 - 3i$

Answer: B

First, $|3 + 4i| = 5$ and $\frac{25i}{3+4i} = \frac{25i(3-4i)}{25} = 4 + 3i$. The expression is $5 - (4 + 3i) = 1 - 3i$, and its complex conjugate is $1 + 3i$.

34. If $z = \sqrt{2} \cos\left(\frac{\pi}{12}\right) + i\sqrt{2} \sin\left(\frac{\pi}{12}\right)$, then what is the value of z^3 ? (1)
- A. ✓ $2 + 2i$ B. $\sqrt{2} + \sqrt{2}i$
 C. $2\sqrt{2} + 2\sqrt{2}i$ D. $3\sqrt{2} + 3\sqrt{2}i$

Answer: A

By De Moivre's Theorem, $z^3 = (\sqrt{2})^3 \left(\cos \frac{3\pi}{12} + i \sin \frac{3\pi}{12}\right) = 2\sqrt{2} \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) = 2 + 2i$.

35. Which of the following relations holds for the sequence: -10, -3, 4, 11,...? (1)
- A. $a_n = a_{n-1} - 8$ B. ✓ $a_n = a_{n-1} + 7$
 C. $a_n = a_{n-1} - 7$ D. $a_n = a_{n-1} + 8$

Answer: B

The sequence is an arithmetic progression with common difference $d = -3 - (-10) = 7$. Therefore, the recursive formula is $a_n = a_{n-1} + 7$.

36. What is the sum of $\sum_{n=1}^{30} (-1)^n \left(\frac{1}{n} + \frac{1}{n+1} \right)$? (1)
- A. $-29/30$ B. $29/30$
 C. $30/31$ D. $\checkmark -30/31$

Answer: D

Writing out the sum reveals a telescoping series: $-(1 + 1/2) + (1/2 + 1/3) - (1/3 + 1/4) + \cdots + (1/30 + 1/31)$. All intermediate terms cancel, leaving $-1 + 1/31 = -30/31$.

37. Which of the following is the sum of the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$? (1)
- A. -5 B. -3
 C. $\checkmark 3$ D. 5

Answer: C

This is an infinite geometric series with first term $a = 5$ and common ratio $r = -2/3$. The sum is $S = \frac{a}{1-r} = \frac{5}{1-(-2/3)} = \frac{5}{5/3} = 3$.

38. Which one of the following is a convergent sequence? (1)
- A. $\left\{ \left(\frac{5}{3} \right)^n \right\}$ B. $\checkmark \left\{ \frac{2n}{n+1} \right\}$
 C. $\left\{ \frac{n^2}{n+1} \right\}$ D. $\left\{ \frac{(-1)^n}{3} \right\}$

Answer: B

The sequence $\left\{ \frac{2n}{n+1} \right\}$ converges to $\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2$. The other sequences either diverge to infinity or oscillate without converging.

39. What is the value of the area of the region enclosed by the graph of $f(x) = e^x$ and $g(x) = x$ between the lines $x = -1$ and $x = 1$? (1)
- A. $\checkmark \frac{e^2 - 1}{e}$ B. $e^2 - \frac{1}{e}$
 C. $e^2 - \frac{1}{e} + 2$ D. $e - \frac{1}{e} + 2$

Answer: A

Since $e^x > x$ on $[-1, 1]$, the area is $\int_{-1}^1 (e^x - x) dx = [e^x - x^2/2]_{-1}^1 = (e - 1/2) - (e^{-1} - 1/2) = e - 1/e = \frac{e^2 - 1}{e}$.

40. If $f(0) = -1$, $f(1) = 2$ and $f'(x)$ is continuous on $[0, 1]$, then which of the following is equal to $\int_0^1 f'(x) \sqrt{2 + f(x)} dx$? (1)
- A. $16/3$ B. $\checkmark 14/3$
 C. $8/3$ D. $4/3$

Answer: B

Substituting $u = 2 + f(x)$ gives $du = f'(x) dx$, with limits $u(0) = 2 - 1 = 1$ and $u(1) = 2 + 2 = 4$. Evaluating $\int_1^4 \sqrt{u} du = [\frac{2}{3} u^{3/2}]_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}$.

41. Which of the following is equal to the volume of a solid generated when the region bounded by the graph of $y = 2\sqrt{x+1}$ and x-axis, when $0 \leq x \leq 2$, rotates about the x-axis? (1)
- A. 8π B. $8\pi^2$
C. $\checkmark 16\pi$ D. $16\pi^2$

Answer: C

The volume of revolution is $V = \pi \int_0^2 y^2 dx = \pi \int_0^2 4(x+1) dx = 4\pi[x^2/2 + x]_0^2 = 4\pi(2+2) = 16\pi$.

42. A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t = 0$. If the particle is at position $x = 2$ at a time $t = 0$, what is the position of the particle at $t = 1$? (1)
- A. $\checkmark 6$ B. 9
C. 11 D. 12

Answer: A

Integrating velocity gives position $x(t) = t^3 + 3t^2 + C$. Since $x(0) = 2$, $C = 2$, so $x(1) = 1^3 + 3(1)^2 + 2 = 6$.

43. Which of the following is equal to $\int x(e^x + \sin(x^2)) dx$? (1)
- A. $e^x(x+1) + \frac{1}{2}\sin(x^2) + c$ B. $e^x(x-1) - \frac{1}{2}\sin(x^2) + c$
C. $\checkmark e^x(x-1) - \frac{1}{2}\cos(x^2) + c$ D. $e^x(x-1) + \frac{1}{2}\cos(x^2) + c$

Answer: C

Splitting the integral gives $\int xe^x dx + \int x \sin(x^2) dx$. Using integration by parts for the first term yields $e^x(x-1)$ and substitution $u = x^2$ for the second term yields $-\frac{1}{2}\cos(x^2) + c$.

44. What is the value of $\int_0^{\ln \sqrt{3}} \frac{e^x}{e^{-x} + e^x} dx$? (1)
- A. $\checkmark \frac{1}{2} \ln 2$ B. $\ln 4$
C. $\ln 2$ D. 1

Answer: A

Multiplying the numerator and denominator by e^x gives $\int_0^{\ln \sqrt{3}} \frac{e^{2x}}{1+e^{2x}} dx = \left[\frac{1}{2} \ln(1+e^{2x})\right]_0^{\ln \sqrt{3}} = \frac{1}{2}(\ln 4 - \ln 2) = \frac{1}{2} \ln 2$.

45. Suppose $f : (-\infty, \infty) \rightarrow \mathbb{R}$ is differentiable and the graph of its derivative $y = f'(x)$, is as shown in the figure below. Which one of the following is true about f ? (1)
- A. f is increasing on $(1, \infty)$ B. f is concave upward on $(0, \infty)$
C. f has no relative maximum value D. $\checkmark f$ has a relative minimum value at $x = 2$

Answer: D

46. If $2 \leq f'(x) \leq 4$ for all values of x , then the value of $f(8) - f(2)$ is between which of the following numbers? (1)
- A. 14 and 24 B. $\checkmark 12$ and 24
C. 12 and 18 D. 8 and 10

Answer: B

By the Mean Value Theorem, $f(8) - f(2) = f'(c)(8-2) = 6f'(c)$ for some $c \in (2, 8)$. Since $2 \leq f'(x) \leq 4$, $12 \leq f(8) - f(2) \leq 24$.

47. A tin can of volume $54\pi \text{ cm}^3$ is to be made in the form of a right circular cylinder that has both flat top and flat bottom. What is the base radius of the tin if it is to be made of the least amount of metal? (1)
- A. 2 cm
B. ✓ 3 cm
C. 4 cm
D. 6 cm

Answer: B

Volume $V = \pi r^2 h = 54\pi \implies h = 54/r^2$. The surface area is $A(r) = 2\pi r^2 + 2\pi r h = 2\pi(r^2 + 54/r)$; setting $A'(r) = 2\pi(2r - 54/r^2) = 0$ gives $r^3 = 27$, so $r = 3 \text{ cm}$.

48. Air is being pumped into a spherical balloon so that its volume increase at a rate of $50 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 5 cm? (1)
- A. $\frac{1}{50\pi} \text{ cm/s}$
B. $\frac{1}{25\pi} \text{ cm/s}$
C. $\frac{5}{\pi} \text{ cm/s}$
D. ✓ $\frac{2}{\pi} \text{ cm/s}$

Answer: D

The volume of a sphere is $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Given the diameter is 5 cm, the radius is $r = 2.5 \text{ cm}$, giving $50 = 4\pi(2.5)^2 \frac{dr}{dt} = 25\pi \frac{dr}{dt}$, which yields $\frac{dr}{dt} = \frac{2}{\pi} \text{ cm/s}$.

49. Which one of the following is a valid assertion that can be proved by principle of mathematical induction? (1)
- A. the sum of any two positive rational numbers is positive.
B. $r^2 \geq 1$, for every real number $r \geq 1$
C. ✓ $n^2 \geq 4n$, for every integer $n \geq 4$
D. $2^2 \leq 2^2$, for every integer $n \leq 100$

Answer: C

The principle of mathematical induction is used to prove statements indexed by integers $n \geq n_0$. Here, $n^2 \geq 4n$ for all integers $n \geq 4$ is an assertion about integers that has a base step $n = 4$ and an inductive step.

50. Consider the assertion: "The sum of positive irrational numbers is positive irrational number". Which one of the following is correct about the assertion? (1)
- A. taking irrational numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}$, and so on, if we add any two of them, the sum is irrational. Therefore, the assertion is true.
B. the sum of $1 + \sqrt{2}$ and $1 - \sqrt{2}$ is 2, which is rational. This is a counter example that disproves the assertion.
C. the sum of $\sqrt{7}$ and $\sqrt{2}$ is a counter example that shows the assertion is false.
D. ✓ the assertion can be disproved by taking the sum of $1 + \sqrt{2}$ and $2 - \sqrt{2}$ as a counter example.

Answer: D

Both $1 + \sqrt{2}$ and $2 - \sqrt{2}$ are positive irrational numbers (since $\sqrt{2} \approx 1.414 < 2$), but their sum is 3, which is rational, providing a valid counterexample. Option B fails because $1 - \sqrt{2}$ is negative.

51. Suppose "If $x \in A$, then $y \in B$ " is a True statement. Then, which one of the following is necessarily true? (1)
- A. $y \in B$
B. if $x \notin A$, then $y \notin B$
C. if $y \in B$, then $x \in A$
D. ✓ if $y \notin B$, then $x \notin A$

Answer: D

The contrapositive of a conditional statement $P \implies Q$ is $\neg Q \implies \neg P$, which is logically equivalent to the original statement. Thus, if $y \notin B$, then $x \notin A$ must be true.

52. Let $\vec{A}$ and $\vec{B}$ be vectors in space such that $\vec{A} \cdot \vec{B} = -2$ and $\vec{B} = 6\vec{i} - 7\vec{j} + \sqrt{15}\vec{k}$. If θ is the angle between $\vec{A}$ and $\vec{B}$, then what is the value of $|\vec{A}|$? (1)
- A. $\frac{1}{5} \cos \theta$ B. $\frac{1}{5 \cos \theta}$
 C. $-\frac{1}{5} \cos \theta$ D. $\frac{-1}{5 \cos \theta}$

Answer: D

The magnitude of $\vec{B}$ is $|\vec{B}| = \sqrt{6^2 + (-7)^2 + (\sqrt{15})^2} = \sqrt{36 + 49 + 15} = 10$. Using the dot product formula $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta$, we get $-2 = 10|\vec{A}| \cos \theta$, which implies $|\vec{A}| = \frac{-1}{5 \cos \theta}$.

53. If $P(2, \sqrt{5}, 1)$ and $Q(3, 0, 9)$ are points on a sphere whose center is on z-axis, then which one of the following point is outside of the sphere? (1)
- A. $(-4, 3, 5)$ B. $(2, -2, 1)$
 C. $(3, 1, 1)$ D. $(0, 0, 0)$

Answer: C

Let the center on the z-axis be $(0, 0, c)$. Equating distances to P and Q gives $4 + 5 + (1 - c)^2 = 9 + (9 - c)^2$, which solves to $c = 5$ and radius $R = 5$. The squared distance from $(3, 1, 1)$ to $(0, 0, 5)$ is $3^2 + 1^2 + (1 - 5)^2 = 26 > 25$, so it lies outside the sphere.

54. If $A(x, 0, 2)$, $B(3, 0, 2)$ and $C(2, \sqrt{3}, 2)$ are vertices of an equilateral triangle in space, then what is the value of x? (1)
- A. 5 B. 3
 C. 2 D. $\sqrt{1}$

Answer: D

The distance $BC = \sqrt{(3 - 2)^2 + (0 - \sqrt{3})^2} = 2$. For an equilateral triangle, $AB = 2 \implies |x - 3| = 2$ ($x = 1$ or 5) and $AC = 2 \implies (x - 2)^2 + 3 = 4$ ($x = 1$ or 3), so the only consistent value is $x = 1$.

55. A patrol boat on a sea sailed from its station 7km to the North; and changed its course and sailed $4\sqrt{2}$ km in the direction of 45° South-East. What is the shortest (straight) distance the boat should travel in order to return to its station? (1)
- A. $\sqrt{5}$ km B. 7km
 C. $5\sqrt{2}$ km D. $5 + \sqrt{2}$ km

Answer: A

Starting at the origin $(0, 0)$, sailing 7 km North puts the boat at $(0, 7)$. Moving $4\sqrt{2}$ km at 45° South-East shifts position by $\Delta x = 4\sqrt{2} \cos 45^\circ = 4$ and $\Delta y = -4\sqrt{2} \sin 45^\circ = -4$, ending at $(4, 3)$, whose distance from the origin is $\sqrt{4^2 + 3^2} = 5$ km.

56. Let $\vec{PQ}$ be a vector with initial point $P = (1, 5)$ and terminal point $Q = (4, 0)$. If $\vec{v} = x\vec{i} + 2\vec{j}$ is parallel to $\vec{PQ}$, then what is the value of x? (1)
- A. $-\frac{6}{5}$ B. $-\frac{2}{5}$
 C. -3 D. 3

Answer: A

The vector $\vec{PQ} = (4 - 1)\vec{i} + (0 - 5)\vec{j} = 3\vec{i} - 5\vec{j}$. Since $\vec{v} = x\vec{i} + 2\vec{j}$ is parallel to $\vec{PQ}$, the components are proportional: $\frac{x}{3} = \frac{2}{-5}$, which gives $x = -\frac{6}{5}$.

