

Mathematics, 2011 EC, entrance exam

Grade 12 · Mathematics · 2011 E.C.

MARKING KEY — CONFIDENTIAL

Time allowed: 120 minutes

Questions: 64

Total marks: 64

1. Which one of the following is the set of critical numbers of $f(x) = \frac{3}{8}x^{8/3} - 6x^{2/3}$? (1)
- A. $\{-2, 2\}$ B. $\{-1, 1\}$
 C. $\{-1, 0, 1\}$ D. ✓ $\{-2, 0, 2\}$

Answer: D

The critical numbers of a function are where its derivative is zero or undefined. By differentiating, we get $f'(x) = \frac{x^2-4}{x^{1/3}}$, which is zero at $x = \pm 2$ and undefined at $x = 0$, all of which are in the domain of f .

2. What is the greatest lower bound of the sequence $\left\{(-1)^n \frac{1}{n+1}\right\}_{n=0}^{\infty}$? (1)
- A. -1 B. 0
 C. ✓ $-1/2$ D. $1/2$

Answer: C

Listing the first few terms of the sequence yields $\{1, -1/2, 1/3, -1/4, 1/5, -1/6, \dots\}$. The negative terms are increasing towards 0, with the absolute minimum and greatest lower bound being the first negative term, which is $-1/2$.

3. Which one of the following is true about the pair of lines: $3x + 9y - 24 = 0$ and $4x + 12y + 32 = 0$? (1)
- A. perpendicular lines B. intersecting lines
 C. ✓ parallel and distinct lines D. representing the same lines

Answer: C

Both equations can be rewritten in slope-intercept form to find their slopes: $y = -\frac{1}{3}x + \frac{8}{3}$ and $y = -\frac{1}{3}x - \frac{8}{3}$. Since they have the same slope but different y -intercepts, they represent parallel and distinct lines.

4. Which one of the following is equal to $f(x) = \sqrt{(x+4)^2}$, for every $x \in \mathbb{R}$? (1)
- A. $g(x) = x + 4$ B. $g(x) = x + 2$
 C. ✓ $g(x) = |x + 4|$ D. $g(x) = |x| + 4$

Answer: C

By definition, for any real number u , $\sqrt{u^2} = |u|$. Substituting $u = x + 4$ gives $g(x) = |x + 4|$.

5. The center of a circle on the line $y = 2x$ and the line $x = 1$ is tangent to the circle at $(1, 6)$. How long is the radius of the circle? (1)
- A. 5 B. 4
 C. 3 D. ✓ 2

Answer: D

The tangent line $x = 1$ is vertical, meaning the radius connecting the center $(h, 2h)$ to the tangent point $(1, 6)$ must be horizontal, so $2h = 6 \implies h = 3$. The radius is the horizontal distance between $(3, 6)$ and $(1, 6)$, which is $|3 - 1| = 2$.

6. What is the value of $\lim_{n \rightarrow \infty} \left(\frac{3^n + 2^n}{6^n} \right)$? (1)
- A. 1
B. ✓ 0
C. ∞
D. $\frac{5}{6}$

Answer: B

We can split the expression into $\lim_{n \rightarrow \infty} [(1/2)^n + (1/3)^n]$. Since both bases are between -1 and 1 , their limits as $n \rightarrow \infty$ are both 0 , making the overall limit 0 .

7. If $F(x) = \int_0^x e^{-t} dt$, then what is the value of $F'(x)$? (1)
- A. ✓ e^{-x}
B. $-e^{-x} - 1$
C. $\frac{e^{-x+1}}{-x+1}$
D. $-e^{-x}$

Answer: A

By the Fundamental Theorem of Calculus, if $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$. Applying this to $f(t) = e^{-t}$ directly yields $F'(x) = e^{-x}$.

8. If $\neg p \Rightarrow r$ is False and $p \Leftrightarrow q$ is True, then which of the following is True? (1)
- A. $p \vee (\neg q \wedge r)$
B. ✓ $\neg p \wedge (q \Rightarrow r)$
C. $\neg p \Rightarrow (q \vee r)$
D. $p \Leftrightarrow (\neg q \vee r)$

Answer: B

For $\neg p \Rightarrow r$ to be false, we must have $\neg p$ as True (p is False) and r as False. Since $p \Leftrightarrow q$ is True, q must also be False; substituting these truth values into $\neg p \wedge (q \Rightarrow r)$ gives $\text{True} \wedge (\text{False} \Rightarrow \text{False}) = \text{True} \wedge \text{True} = \text{True}$.

9. If $a_1 = 2, a_2 = 6, a_3 = 10, a_4 = 14, \dots$, then $\sum_{n=1}^{100} a_n = ___$ (1)
- A. 20,020
B. 20,200
C. ✓ 20,000
D. 22,000

Answer: C

This is an arithmetic sequence with first term $a_1 = 2$ and common difference $d = 4$. Using the sum formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ with $n = 100$, we get $S_{100} = 50[4 + 99(4)] = 20,000$.

10. What is the value of $\lim_{h \rightarrow 0} \frac{8(\frac{1}{2}+h)^2 - 8(\frac{1}{2})^2}{h}$? (1)
- A. 0
B. 4
C. ✓ 8
D. The limit does not exist

Answer: C

This limit represents the definition of the derivative of $f(x) = 8x^2$ evaluated at $x = 1/2$. Since $f'(x) = 16x$, the value of the limit is $f'(1/2) = 16(1/2) = 8$.

11. If there are two children in a family, what is the probability that there is at least one girl in the family? (1)
- A. $\frac{1}{2}$
B. $\frac{1}{4}$
C. ✓ $\frac{3}{4}$
D. $\frac{2}{3}$

Answer: C

The sample space for a family with two children is $\{BB, BG, GB, GG\}$, where each outcome is equally likely. The event of having 'at least one girl' contains three of these outcomes (BG, GB, GG), giving a probability of $3/4$.

12. If $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 2 & x \\ 2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -x & 3 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & -2 \end{vmatrix}$, then what is the value of x ? (1)
- A. $\checkmark \frac{2}{5}$ B. $-\frac{2}{5}$
 C. $\frac{2}{3}$ D. $-\frac{2}{3}$

Answer: A

Evaluating the determinants on both sides gives the equation $6x + 11 = x + 13$. Solving for x yields $5x = 2$, which simplifies to $x = 2/5$.

13. Let $f(x) = 2e^x - k \sin x + 1$. If the equation of the tangent line to the graph of f at $(0, 3)$ is $y = 5x + 3$, then what is the value of k ? (1)
- A. 3 B. $\checkmark -3$
 C. -5 D. 2

Answer: B

The slope of the tangent line to $f(x)$ at $x = 0$ is $f'(0) = 5$. Since $f'(x) = 2e^x - k \cos x$, we have $2(1) - k(1) = 5$, which yields $k = -3$.

14. Everyday a person saves 5 cents more than the amount he saved on the previous day. His target is to save the total amount of 3225 cents by the end of 30 days. How much must be the starting saving to meet the target? (1)
- A. $\checkmark$ 35 cents B. 25 cents
 C. 50 cents D. 60 cents

Answer: A

This is an arithmetic series with $d = 5$, $n = 30$, and $S_{30} = 3225$. Using the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$, we get $3225 = 15[2a_1 + 145]$, which simplifies to $215 = 2a_1 + 145$, so the starting saving $a_1 = 35$ cents.

15. Which one of the following is the multiplicative inverse of $z = \frac{3+4i}{4-5i}$? (1)
- A. $\frac{8}{25} - \frac{31}{25}i$ B. $-\frac{8}{25} + \frac{31}{25}i$
 C. $\checkmark -\frac{8}{25} - \frac{31}{25}i$ D. $\frac{8}{25} + \frac{31}{25}i$

Answer: C

The multiplicative inverse of z is $1/z = \frac{4-5i}{3+4i}$. Multiplying the numerator and denominator by the complex conjugate $3 - 4i$ gives $\frac{(4-5i)(3-4i)}{25} = \frac{-8-31i}{25} = -\frac{8}{25} - \frac{31}{25}i$.

16. Consider the following system of equations:
$$\begin{cases} x - 2y + z = 1 \\ -x + y + z = 3 \\ 3x - 5y + z = k \end{cases}$$
 (1)
- How much must be the value of k so that the system has a solution?
- A. 7 B. 1
 C. 0 D. $\checkmark -1$

Answer: D

Using Gaussian elimination on the augmented matrix of the system, Row 3 - 3*(Row 1) + (Row 2 + Row 1) leads to a bottom row of $[0, 0, 0|k + 1]$. For a consistent solution, we must have $k + 1 = 0$, giving $k = -1$.

17. What is the value of $\int_1^2 \frac{\ln x}{x^2} dx$? (1)
- A. $-\frac{\ln 2}{2} - \frac{1}{2}$ B. $\frac{\ln 2}{2} - \frac{1}{2}$
 C. $\frac{\ln 2}{2} + \frac{1}{2}$ D. $-\frac{\ln 2}{2} + \frac{1}{2}$

Answer: C

Using integration by parts with $u = \ln x$ and $dv = x^{-2} dx$, we find the antiderivative is $-\frac{\ln x}{x} - \frac{1}{x}$. Evaluating this from 1 to 2 yields $(-\frac{\ln 2}{2} - \frac{1}{2}) - (0 - 1) = -\frac{\ln 2}{2} + \frac{1}{2}$.

18. Among 2000 students who took a regional exam, the percentile of certain student's score is 90. Which of the following is correct about the student's score? (1)
- A. The student have answered 90% of the questions correctly.
 B. **The student's score is greater than or equal to that of 1800 students.**
 C. The student's score is the same as the top 10% of the scores.
 D. The score of the student is as good as that of 90% of the students.

Answer: B

A 90th percentile means that 90% of the scores are less than or equal to the student's score. Since there are 2000 students, 90% of 2000 = 1800 students scored less than or equal to this student.

19. If A is a 3×3 matrix and $\det(A) = 5$, then how much is $\det(2A^T A)$? (1)
- A. **200** B. 100
 C. 50 D. 20

Answer: A

Using determinant properties, since A is a 3×3 matrix, $\det(2A^T A) = 2^3 \det(A^T) \det(A) = 8 \det(A)^2 = 8(5^2) = 200$.

20. What is the simplified form of $\frac{a^{-1}b^{-1}}{a^{-3}-b^{-3}}$? (1)
- A. $\frac{a^2b^2}{b^2-a^2}$ B. **$\frac{a^2b^2}{b^3-a^3}$**
 C. $\frac{a^3-b^3}{ab}$ D. $\frac{a^3-b^3}{a-b}$

Answer: B

Writing the exponents as fractions gives $\frac{1/(ab)}{1/a^3-1/b^3} = \frac{1/(ab)}{(b^3-a^3)/(a^3b^3)} = \frac{a^3b^3}{ab(b^3-a^3)} = \frac{a^2b^2}{b^3-a^3}$.

21. If the circle passing through the point $(-1, 0)$ touches the y -axis at $(0, 2)$, then what is the equation of the circle? (1)
- A. $x^2 + y^2 + 5x + 4y + 4 = 0$ B. $x^2 + y^2 - 5x + 4y + 4 = 0$
 C. $x^2 + y^2 - 5x - 4y + 4 = 0$ D. **$x^2 + y^2 + 5x - 4y + 4 = 0$**

Answer: D

A circle touching the y -axis at $(0, 2)$ has a center at $(h, 2)$ and radius $r = |h|$. Since it passes through $(-1, 0)$, we have $(h+1)^2 + 4 = h^2$, which gives $h = -5/2$. The circle's equation is $(x+5/2)^2 + (y-2)^2 = 25/4$, simplifying to $x^2 + y^2 + 5x - 4y + 4 = 0$.

22. Let z be a complex number and $w = 3 + 4i$. If $\frac{z^2+1}{z+i} = |w|z - \frac{1}{i}\bar{w}$, then what is the value of z ? (1)
- A. $-4 - 2i$ B. $4 - 2i$
 C. $-1 + i$ D. **$-1 - i$**

Answer: D

Simplifying the LHS, $\frac{z^2+1}{z+i} = z - i$ (for $z \neq -i$). Substituting $|w| = 5$ and $\bar{w} = 3 - 4i$, we get $z - i = 5z - \frac{3-4i}{i} = 5z + (4 + 3i)$, which simplifies to $-4z = 4 + 4i$, yielding $z = -1 - i$.

23. What is the value of the left-hand side limit, $\lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} \sqrt{4x^3 + 9x^2}$? (1)
- A. ✓ -3 B. 3
C. 2 D. Does not exist

Answer: A

Since $x \rightarrow 0^-$, we have $x < 0$, so $\sqrt{4x^3 + 9x^2} = \sqrt{x^2(4x + 9)} = -x\sqrt{4x + 9}$. Thus, the limit becomes $\lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} (-x\sqrt{4x + 9}) = -\lim_{x \rightarrow 0^-} \frac{\sin x}{x} \sqrt{4x + 9} = -1 \times 3 = -3$.

24. What is the value of $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$? (1)
- A. 2 B. ✓ 3
C. -2 D. The limit does not exist.

Answer: B

By factoring the numerator and the denominator, we get $\lim_{x \rightarrow 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3$.

25. If $f(x) = ax^3 + \frac{b}{x} + 5$ has local minimum at $(2, -3)$, what are the values of a and b ? (1)
- A. $a = -\frac{1}{4}$, $b = 12$ B. ✓ $a = -\frac{1}{4}$, $b = -12$
C. $a = \frac{1}{4}$, $b = 12$ D. $a = \frac{1}{4}$, $b = -12$

Answer: B

Since $(2, -3)$ is a point on the curve, substituting $x = 2$ gives $8a + b/2 + 5 = -3$, which simplifies to $16a + b = -16$. Using the fact that there is a local minimum at $x = 2$, we have $f'(2) = 12a - b/4 = 0$, giving $b = 48a$, which yields $a = -1/4$ and $b = -12$ when solved simultaneously.

26. What is the value of $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx$? (1)
- A. $\frac{1}{e} \ln(x^e + e^{x+1}) + c$ B. ✓ $\frac{1}{e} \ln(x^e + e^x) + c$
C. $\frac{1}{e} \ln((x+1)^e + e^{x+1}) + c$ D. $\frac{1}{e} \ln(x^{e+1} + e^{x+1}) + c$

Answer: B

Using the substitution $u = x^e + e^x$, we have $du = (ex^{e-1} + e^x)dx = e(x^{e-1} + e^{x-1})dx$. Thus, the integral becomes $\int \frac{1}{e} \frac{du}{u} = \frac{1}{e} \ln|u| + c = \frac{1}{e} \ln(x^e + e^x) + c$.

27. What is the area of the region bounded by the lines $x = 0$, $x = 2$, and $y = 1$ and the curve $y = e^{2x}$? (1)
- A. ✓ $\frac{e^4}{2} - \frac{5}{2}$ B. $\frac{e^2}{2} - \frac{1}{2}$
C. $\frac{e^3}{3} - \frac{1}{3}$ D. $-\frac{e^4}{2} + \frac{5}{2}$

Answer: A

The area is given by the definite integral of the upper curve minus the lower line, which is $\int_0^2 (e^{2x} - 1)dx = [\frac{1}{2}e^{2x} - x]_0^2 = (\frac{1}{2}e^4 - 2) - (\frac{1}{2}e^0 - 0) = \frac{e^4}{2} - \frac{5}{2}$.

28. If the region enclosed by the graphs of $f(x) = x^2$ and $f(x) = x^3$ from $x = 0$ to $x = 1$ rotates about the x -axis, what is the volume of the solid of revolution? (1)
- A. $\frac{2\pi}{27}$ cubic units B. $\frac{2\pi}{25}$ cubic units
C. $\frac{2\pi}{5}$ cubic units D. ✓ $\frac{2\pi}{35}$ cubic units

Answer: D

Using the washer method, the volume is given by $V = \pi \int_0^1 ([x^2]^2 - [x^3]^2)dx = \pi \int_0^1 (x^4 - x^6)dx = \pi [\frac{x^5}{5} - \frac{x^7}{7}]_0^1 = \frac{2\pi}{35}$.

29. Which one of the following is a valid argument?

(1)

- A. If I don't change my oil regularly, my engine will die. My engine died, Thus, I didn't change my oil regularly.
- B. ✓ **If I am literate, then I can read and write. I can read but I can't write. Thus I am not literate.**
- C. If you do every problem in the book, then you will learn the subject. You learned the subject. Thus, you did every problem in the book.
- D. If it rains or snows, then my roof leaks. My roof is leaking. Thus, it is raining and snowing.

Answer: B

Option B is valid by Modus Tollens: $L \implies (R \wedge W)$, and we are given $R \wedge \neg W$, which implies $\neg(R \wedge W)$, so we can validly conclude $\neg L$. Options A, C, and D are invalid because they commit the fallacy of affirming the consequent.

30. The earth's orbit has a semi-major axis $a \approx 149.6$ Gm (gigameters) and an eccentricity of $e \approx 0.017$. What is the approximate value of the semi-minor axis? (1)

- A. 152.14 Gm
- B. 145.32 Gm
- C. 149.06 Gm
- D. ✓ **149.58 Gm**

Answer: D

The relationship between the semi-major axis a , semi-minor axis b , and eccentricity e is $b = a\sqrt{1 - e^2}$. Calculating with $a = 149.6$ and $e = 0.017$, we get $b \approx 149.6 \times 0.9998555 \approx 149.58$ Gm.

31. The marks of 50 students is given below:

(1)

If the median of the data is 26, what are the values of f_1 and f_2 ?

[Figure: A table with two rows and six columns:

Row 1: "Marks", "0-10", "10-20", "20-30", "30-40", "40-50"

Row 2: "No. of students", "5", "8", " f_1 ", "10", " f_2 "]

- A. $f_1 = 7, f_2 = 20$
- B. $f_1 = 12, f_2 = 15$
- C. $f_1 = 15, f_2 = 12$
- D. ✓ $f_1 = 20, f_2 = 7$

Answer: D

With a total of 50 students, we have $5 + 8 + f_1 + 10 + f_2 = 50 \implies f_1 + f_2 = 27$. Using the grouped median formula $\text{Median} = L + \left(\frac{N/2 - CF}{f}\right)w$, we get $26 = 20 + \left(\frac{25 - 13}{f_1}\right)10 \implies f_1 = 20$, which yields $f_2 = 7$.

32. Suppose 2,500 items are produced by a machine and 2 % of the product are randomly selected and tested. If 5 of the tested items have a defect, then what is the probability that an item produced by the machine has NO defect? (1)

- A. 0.80
- B. 0.85
- C. ✓ **0.90**
- D. 0.95

Answer: C

The size of the tested sample is 2% of 2500, which is 50 items. Since 5 of these tested items are defective, the estimated probability of a defective item is $5/50 = 0.10$, meaning the probability of a non-defective item is $1 - 0.10 = 0.90$.

33. If $2 \begin{pmatrix} 2x & x - 5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & -5 & -4 \end{pmatrix}$, then what is the value of x ? (1)

- A. -2
- B. ✓ **2**
- C. $\frac{1}{2}$
- D. $-\frac{1}{2}$

Answer: B

Letting A be the matrix, we have $2A^{-1} = B \implies A = 2B^{-1}$. Calculating the inverse of $B = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$ gives $B^{-1} = \frac{1}{-2} \begin{pmatrix} -4 & -2 \\ 5 & 3 \end{pmatrix}$, so $A = \begin{pmatrix} 4 & 2 \\ -5 & -3 \end{pmatrix}$, which means $2x = 4 \implies x = 2$.

34. If the second and fifth terms of a geometric progression are $-\frac{1}{2}$ and $\frac{1}{16}$, respectively, what is the sum of the first eight terms of the sequence? (1)
- A. $\checkmark \frac{85}{128}$ B. $\frac{255}{256}$
 C. $\frac{128}{85}$ D. $\frac{256}{255}$

Answer: A

In a geometric progression, $ar = -1/2$ and $ar^4 = 1/16$, which gives $r^3 = -1/8 \implies r = -1/2$ and $a = 1$. The sum of the first eight terms is $S_8 = \frac{a(1-r^8)}{1-r} = \frac{1-(1/256)}{1.5} = \frac{255/256}{3/2} = \frac{85}{128}$.

35. What is the limit of the sequence $1, \frac{2}{2^2-1^2}, \frac{3}{3^2-2^2}, \frac{4}{4^2-3^2}, \dots$? (1)
- A. 0 B. 1
 C. $\checkmark \frac{1}{2}$ D. $\frac{3}{2}$

Answer: C

The n -th term of the sequence is $a_n = \frac{n}{n^2-(n-1)^2} = \frac{n}{2n-1}$. Taking the limit of this sequence as $n \rightarrow \infty$, we find $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2}$.

36. At what value of x does the function $f(x) = \frac{4x^3}{3} - x^4$ attain its maximum value? (1)
- A. -1 B. 0
 C. $\checkmark 1$ D. $\frac{4}{3}$

Answer: C

The derivative of $f(x)$ is $f'(x) = 4x^2 - 4x^3 = 4x^2(1-x)$. The critical numbers are $x = 0$ and $x = 1$, and since $f'(x) > 0$ for $x < 1$ (except at 0) and $f'(x) < 0$ for $x > 1$, the function achieves its maximum at $x = 1$.

37. What is the solution of $\int 4x(\ln x + \frac{1}{x^2}) dx$? (1)
- A. $4x^2(\ln x + 1) - 2x^2 + c$ B. $x^2(4 \ln x - 1) + 2 \ln x + c$
 C. $x^2(2 \ln x + 1) + 4 \ln x + c$ D. $\checkmark x^2(2 \ln x - 1) + 4 \ln x + c$

Answer: D

The integral can be split into $\int 4x \ln x dx + \int \frac{4}{x} dx$. Using integration by parts on the first term yields $2x^2 \ln x - x^2 + c_1$, while the second term integrates to $4 \ln x + c_2$, giving a combined result of $x^2(2 \ln x - 1) + 4 \ln x + c$.

38. If $f(x) = ax - b$ and $f^{-1}(x+1) = \frac{1}{2}x + 2$, for each $x \in \mathbb{R}$, then what must be the values of a and b ? (1)
- A. $a = 1/2, b = -2$ B. $a = 2, b = 2$
 C. $a = 1, b = 1$ D. $\checkmark a = 2, b = 3$

Answer: D

Using the relation $f^{-1}(y) = z \iff f(z) = y$, we have $f(\frac{1}{2}x + 2) = x + 1$. Substituting this into $f(t) = at - b$ gives $a(\frac{1}{2}x + 2) - b = x + 1$, which yields $a = 2$ and $b = 3$ upon comparing coefficients.

39. Which one of the following is true about the graph of $f(x) = \frac{x^2+5x+6}{x^2-4} + 3$? (1)
- A. The graph has a hole at $x = 2$.
 B. The vertical asymptotes of the graph are $x = -2$ and $x = 2$.
 C. $\checkmark$ A horizontal asymptote of the graph is $y = 4$.
 D. The graph has y -intercept at $(0, -\frac{3}{2})$

Answer: C

Simplifying the rational expression gives $f(x) = \frac{(x+2)(x+3)}{(x+2)(x-2)} + 3 = \frac{x+3}{x-2} + 3$ for $x \neq -2$. As $x \rightarrow \infty$, $f(x) \rightarrow 1 + 3 = 4$, making $y = 4$ a horizontal asymptote.

40. Fatuma can solve 90% of the problems given in a book and Mesfin can solve 70%. What is the probability that at least one of them will solve the problem, selected at random from the book? (1)
- A. 0.77
B. 0.87
C. ✓ 0.97
D. 0.67

Answer: C

The probability of at least one solving the problem is $1 - P(\text{neither solves}) = 1 - (1 - 0.90)(1 - 0.70) = 1 - 0.03 = 0.97$.

41. What is the polar form of $\frac{7-i}{3-4i}$? (1)
- A. $\sqrt{2}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$
B. ✓ $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$
C. $2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$
D. $2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$

Answer: B

Multiplying the numerator and denominator by the conjugate $3 + 4i$ simplifies the expression to $1 + i$. Converting this complex number to polar form gives a modulus of $\sqrt{1^2 + 1^2} = \sqrt{2}$ and argument of $\arctan(1) = \pi/4$, resulting in $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$.

42. What is the sum of the infinite series $\sum_{k=0}^{\infty} 5 \left(\frac{2^k + 5^k}{10^k} \right)$? (1)
- A. 15
B. $\frac{25}{4}$
C. 5
D. ✓ $\frac{65}{4}$

Answer: D

Splitting the sum into two infinite geometric series, we have $5 \sum_{k=0}^{\infty} (1/5)^k + 5 \sum_{k=0}^{\infty} (1/2)^k = 5 \left(\frac{1}{1-1/5} \right) + 5 \left(\frac{1}{1-1/2} \right) = \frac{25}{4} + 10 = \frac{65}{4}$.

43. If $f(x) = \ln(2^{\tan x})$, then what is the value of $f'(0)$? (1)
- A. ✓ $\ln 2$
B. $-2 \ln 2$
C. $\frac{\ln 2}{2}$
D. 1

Answer: A

Using properties of logarithms, we can rewrite $f(x) = \ln(2^{\tan x}) = \tan x \ln 2$. Differentiating gives $f'(x) = \sec^2 x \ln 2$, so evaluating at $x = 0$ yields $f'(0) = 1 \cdot \ln 2 = \ln 2$.

44. If $h(x) = \sqrt{1 + \sqrt{x}}$, then which of the following is equal to $h'(x)$? (1)
- A. $\frac{1}{2\sqrt{1+\sqrt{x}}}$
B. ✓ $\frac{1}{4\sqrt{x+x\sqrt{x}}}$
C. $\frac{x}{2\sqrt{1+\sqrt{x}}}$
D. $\frac{x}{4\sqrt{x+x\sqrt{x}}}$

Answer: B

Using the chain rule, $h'(x) = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{d}{dx}(1 + \sqrt{x}) = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}} = \frac{1}{4\sqrt{x+x\sqrt{x}}}$.

- 45.** A man wants to fence two identical rectangular enclosures in a field alongside a straight river as shown in the following figure. (1)

What is the maximum area of each enclosure that he can make with 192 meter fencing material if the side along the river does not need a fence?

[Figure: A diagram shows a straight river represented by a shaded horizontal strip along the top. Directly adjacent to the river is a large rectangular area divided into two identical side-by-side rectangular enclosures by a line perpendicular to the river bank. The fencing consists of the three boundary segments perpendicular to the river and the single outer boundary segment parallel to the river; the side along the river requires no fence.]

- A. 1530 m²
B. 1564 m²
C. 1664 m²
D. ✓ 1536 m²

Answer: D

Let the side perpendicular to the river be x and the side parallel to the river be y for each enclosure. The total fencing length is $3x + 2y = 192$, and the area of each enclosure is $A = xy = x(96 - 1.5x)$. Maximizing this quadratic equation yields $x = 32$ m, which gives a maximum area of 1536 m^2 per enclosure.

- 46.** For real numbers x and y , which one of the following is NOT true? (1)

- A.** $(\forall x)(\forall y)(y^2 + x^2 \geq -1)$ **B.** $(\forall x)(\exists y)(y \geq x^2 + 1)$
C. $\checkmark (\exists x)(\forall y)(y \geq x^2 + 1)$ **D.** $(\exists x)(\exists y)(y \geq x^2 + 1)$

Answer: C

Statement C asserts that there exists a fixed real number x such that for every real number y , $y \geq x^2 + 1$. This is false because for any fixed x , we can always choose a y that is smaller than $x^2 + 1$, such as $y = x^2$.

47. If $f(x) = x^3 - 2x + 1$, then which one of the following is NOT true? (1)

- A. $f(c) = 0$ for some $c \in [-2, 0]$.
 B. $f(c) = \frac{1}{2}$ for some $c \in [0, 1]$.
 C. $\checkmark$ $f(c) = \frac{1}{4}$ for some $c \in [-1, 0]$.
 D. $f(c) = 3$ for some $c \in [1, 2]$.

Answer: C

On the interval $[-1, 0]$, $f(x)$ starts at $f(-1) = 2$, increases to a maximum, and decreases to $f(0) = 1$. Since $f(x) \geq 1$ for all $x \in [-1, 0]$, there is no value of $c \in [-1, 0]$ where $f(c) = 1/4$.

- 48.** A water tank is a rectangular parallelepiped with base length 3 m, width 2 m and height 2.5 m. If water is flowing into the tank at the rate of $0.12 \text{ m}^3/\text{sec}$, then how fast does the level of water rise up in the tank? (1)

- A. ✓ 0.02 m/sec B. 0.03 m/sec
C. 0.04 m/sec D. 0.06 m/sec

Answer: A

The volume of water in the tank is $V = L \cdot W \cdot h = 6h$. Differentiating both sides with respect to time t gives $\frac{dV}{dt} = 6\frac{dh}{dt}$. Substituting $\frac{dV}{dt} = 0.12$ gives $\frac{dh}{dt} = \frac{0.12}{6} = 0.02$ m/sec.

- 49.** If f is the greatest integer function and g is the absolute value function, then what is the value of $(f \circ g)(\frac{3}{2}) + (g \circ f)(-\frac{4}{3})$? (1)

- A. 1
C. $-1/2$
- B. ✓ 3
D. 2

Answer: B

We calculate $(f \circ g)(\frac{3}{2}) = f(\lceil 1.5 \rceil) = \lfloor 1.5 \rfloor = 1$ and $(g \circ f)(-\frac{4}{3}) = g(\lfloor -1.33... \rfloor) = g(-2) = \lceil -2 \rceil = 2$. Their sum is $1 + 2 = 3$.

50. There are three children in a room with, ages four, five, and six. If a five-year-old child enters the room, then which of the following statement is correct ? (1)

- A. ✓ **mean age will stay the same but the standard deviation will decrease.**
B. mean age will stay the same but the standard deviation will increase.
C. mean age and standard deviation will increase.
D. mean age and standard deviation will stay the same.

Answer: A

The initial ages are 4, 5, 6, which have a mean of 5. Adding another 5-year-old keeps the mean at 5, but because the new data point is equal to the mean, it reduces the overall spread, thereby decreasing the standard deviation.

51. If $f(x) = \frac{1}{x}$, then what is the value of $f^{(n)}(x)$? (1)

- A. $f^{(n)}(x) = \frac{(-1)^n n!}{x^n}$
B. $f^{(n)}(x) = \frac{(-1)^{n+1} (n+1)!}{x^{n+1}}$
C. $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n-1}}$
D. ✓ $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

Answer: D

By taking successive derivatives of $f(x) = x^{-1}$, we get $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, and in general, $f^{(n)}(x) = (-1)^n n! x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}$.

52. What is the value of $\cot 270^\circ + 2 \cos 90^\circ + 4 \sec^2 180^\circ$? (1)

- A. -2
B. 8
C. ✓ 4
D. 7

Answer: C

Evaluating each term, we have $\cot 270^\circ = 0$, $\cos 90^\circ = 0$, and $\sec 180^\circ = -1$ so $\sec^2 180^\circ = 1$. The expression simplifies to $0 + 2(0) + 4(1) = 4$.

53. $\forall n \in \mathbb{N}$, $3^n - 2$ is prime that can be proved or disproved by which one of the following mathematical proofs? (1)

- A. Direct proof
B. Proof by exhaustion
C. ✓ **Disprove by counter example**
D. Proof by contradiction

Answer: C

The statement claims $3^n - 2$ is prime for all natural numbers n . This can be disproved by finding a single counter example, such as $n = 3$, which gives $3^3 - 2 = 25$, which is composite.

54. If the point $(\alpha, 0, 3)$ is on the sphere centered at $(1, 2, 3)$ with radius 2, what is the value of α ? (1)

- A. 0
B. ✓ 1
C. 2
D. -3

Answer: B

The equation of the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 2^2 = 4$. Substituting the point $(\alpha, 0, 3)$ gives $(\alpha - 1)^2 + (0 - 2)^2 + (3 - 3)^2 = 4$, which simplifies to $(\alpha - 1)^2 = 0$, yielding $\alpha = 1$.

55. Consider a rectangle ABCD with base vertices $A=(0,3)$ and $B=(4,0)$ and the other vertices, C and D, in the first-quadrant of the coordinate plane. If its height $\overline{BC}$ is half of the length of its base, then which of the following indicates the coordinates of vertex C? (1)

- A. $(4, 5/2)$ B. $(6, 3/2)$
C. $(5/2, -2)$ D. $(11/2, 2)$

Answer: D

The base vector is $\vec{AB} = (4, -3)$ with length 5. A perpendicular vector pointing into the first quadrant is $(3, 4)$, and since height $BC = 2.5$, the displacement vector is $\vec{BC} = 2.5 \times (3/5, 4/5) = (1.5, 2)$, giving $C = (4, 0) + (1.5, 2) = (11/2, 2)$.

56. Which one of the following is a valid assertion that can be proved by the principle of mathematical induction? (1)

- A. $3n + 25 < 3^n$, for every integer $n \geq 3$. B. $2^n > n + 20$, for every integer $n \geq 4$.
C. $n^2 \leq 2^n$, for every integer $n \geq 1$. D. $n^3 - n$ is divisible by 6, for every integer $n \geq 1$
✓ .

Answer: D

The expression $n^3 - n = n(n-1)(n+1)$ is the product of three consecutive integers, meaning at least one is divisible by 3 and at least one is even, making the product always divisible by 6, which is easily proven by mathematical induction. The other statements are false for small values of n (e.g., $n = 3$ for A, $n = 4$ for B, and $n = 3$ for C).

57. Ship A and B depart from the same point at the same time on the course $N60^\circ E$ and $N40^\circ E$, respectively. If the speed of ship A is 20 km per hour and the speed of ship B is 30 km per hour, what is the distance between the two ships just after 30 minutes of their departure? [You may take: $\cos(40^\circ) = 0.77$, $\cos(20^\circ) = 0.94$, $\sin(20^\circ) = 0.34$] (1)

- A. $\sqrt{40}$ km B. $\sqrt{43}$ km
C. $\sqrt{50}$ km D. $\sqrt{53}$ km

Answer: B

After 30 minutes, Ship A travels 10 km and Ship B travels 15 km, with an angle of 20° between their paths. Using the Law of Cosines, the distance squared is $10^2 + 15^2 - 2(10)(15)\cos(20^\circ) = 325 - 300(0.94) = 43$, so the distance is $\sqrt{43}$ km.

58. When the plane is rotated 45° about the point $(1, -2)$, then what would be the image of the point $(2, 4)$? (1)

- A. $\left(1 - \frac{5\sqrt{2}}{2}, -2 + \frac{7\sqrt{2}}{2}\right)$ B. $\left(1 - \frac{\sqrt{2}}{2}, -2 + \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{5\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$ D. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Answer: A

First, translate the center of rotation to the origin, giving the point $(2-1, 4-(-2)) = (1, 6)$. Rotating this by 45° gives $(1 \cos 45^\circ - 6 \sin 45^\circ, 1 \sin 45^\circ + 6 \cos 45^\circ) = \left(-\frac{5\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$, and translating back to the center $(1, -2)$ yields $\left(1 - \frac{5\sqrt{2}}{2}, -2 + \frac{7\sqrt{2}}{2}\right)$.

59. If $\mathbf{u} = 2\vec{j} - \vec{k}$ and $\mathbf{v} = \vec{i} - 8\vec{j} + 3\vec{k}$, then what is the unit vector in the direction of $5\mathbf{u} + \mathbf{v}$? (1)

- A. $\vec{i} + 2\vec{j} - 2\vec{k}$ B. $\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$
C. $\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$ D. $\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

Answer: C

The vector $5\mathbf{u} + \mathbf{v} = 5(2\vec{j} - \vec{k}) + (\vec{i} - 8\vec{j} + 3\vec{k}) = \vec{i} + 2\vec{j} - 2\vec{k}$. The magnitude of this vector is $\sqrt{1^2 + 2^2 + (-2)^2} = 3$, so the unit vector is $\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$.

60. The diagram below is a representation of a 25 m vertical observation tower TB and two cars K and L on a road. (1)
The angle of depression from T to car L is 30° . The angle of elevation from car K to the top of the tower is 60° . B, K and L lie in a straight line and lie on the same horizontal plane as the base of the tower.
What is the distance between the two cars?

[Figure: A diagram showing a vertical cylindrical observation tower TB of height 25 m with base B and top T, standing perpendicular to a horizontal ground line containing points B, K, and L in order from left to right. A right-angle symbol is marked at base B. A horizontal dashed line extends to the right from top T. A straight line of sight runs from T to car L, forming an angle of depression labeled 30° with the horizontal dashed line. Another line runs from car K to top T, forming an angle of elevation labeled 60° with the horizontal line BL.]

- A. $\checkmark \frac{50\sqrt{3}}{3}$ m
B. $50\sqrt{3}$ m
C. $\frac{50\sqrt{3}}{2}$ m
D. $50 + \sqrt{3}$ m

Answer: A

Using trigonometry, the distance $BL = \frac{25}{\tan 30^\circ} = 25\sqrt{3}$ m and $BK = \frac{25}{\tan 60^\circ} = \frac{25\sqrt{3}}{3}$ m. Since K lies between B and L, the distance between the two cars is $KL = BL - BK = 25\sqrt{3} - \frac{25\sqrt{3}}{3} = \frac{50\sqrt{3}}{3}$ m.

61. Let L be the line given by the vector equation $(x, y) = (1, 1) + t(\sqrt{3}, 1)$, $t \in \mathbb{R}$. What is the equation of the image of L after being rotated 15° about $(1, 1)$ and then translated by vector $\mathbf{u} = (-1, 1)$? (1)

- A. $\checkmark -x + y = 2$
B. $x - y = 2$
C. $\sqrt{3}x - y = 2$
D. $-x + \sqrt{3}y = 1$

Answer: A

The line initially has a direction angle of 30° because its direction vector is $(\sqrt{3}, 1)$. After rotation of 15° about $(1, 1)$, the new angle is 45° , giving the line $y = x$. Translating this line by $\mathbf{u} = (-1, 1)$ shifts the coordinate variables to $x' = x - 1$ and $y' = y + 1$, which yields $y' - 1 = x' + 1$, or $-x + y = 2$.

62. If the angle between the vectors $\vec{A} = (2, -1, 1)$ and $\vec{B} = (1, 1, \alpha)$ is $\frac{\pi}{3}$, then what is the value of α ? (1)
A. 1
B. -1
C. $\checkmark$ 2
D. -2

Answer: C

Using the dot product formula $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos(\frac{\pi}{3})$, we get $2 - 1 + \alpha = \sqrt{6}\sqrt{2 + \alpha^2} \cdot \frac{1}{2}$. Solving $2(\alpha + 1) = \sqrt{6(\alpha^2 + 2)}$ yields $(\alpha - 2)^2 = 0$, so $\alpha = 2$.

63. What is the solution set of $\sin^2 x - \sin x \cos x = 0$ over $[0, 2\pi]$? (1)
A. $\{0, \pi, \frac{5\pi}{4}, 2\pi\}$
B. $\{0, \frac{\pi}{4}, \pi, 2\pi\}$
C. $\checkmark \{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi\}$
D. $\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}\}$

Answer: C

Factoring the equation gives $\sin x(\sin x - \cos x) = 0$, meaning either $\sin x = 0$ or $\tan x = 1$. Over the closed interval $[0, 2\pi]$, this yields the solutions $x \in \{0, \pi, 2\pi\}$ and $x \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$ respectively, giving the complete set $\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi\}$.

64. Which of the following is a correct assertion that can be proved by the principle of mathematical induction? (1)
A. $\frac{1}{n+1} \leq 1$, for each real number $n \geq 1$.
B. $m! \leq 4^m$, for each positive integer m .
C. $2^p - 1$ is prime for each prime integer p .
D. $\checkmark k! \geq 2^k$, for each integer $k \geq 4$.

Answer: D

The statement $k! \geq 2^k$ is true for the base case $k = 4$ ($24 \geq 16$), and the inductive step $(k+1)! = (k+1)k! \geq (k+1)2^k \geq 2 \cdot 2^k = 2^{k+1}$ holds for all $k \geq 4$, making it provable by mathematical induction. Options B and C are false for $m = 9$ and $p = 11$ respectively, and option A concerns real numbers rather than integers.

— END OF MARKING KEY —

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