

Mathematics, 2015 EC, entrance exam, version 1

Grade 12 · Mathematics · 2015 E.C.

MARKING KEY — CONFIDENTIAL

Time allowed: 120 minutes

Questions: 57

Total marks: 57

1. If $f(x) = \frac{\sqrt{x+2}}{x+2}$ and $g(x) = \frac{1}{x} - 2$ then what is the value of $f(g(x))$ for all $x > 0$? (1)
- A. $\sqrt{x} - 2$ B. $\sqrt{x} + 2$
 C. $\sqrt{x}$ D. $1/\sqrt{x}$

Answer: C

Substituting $g(x) = \frac{1}{x} - 2$ into $f(x)$ gives $f(g(x)) = \frac{\sqrt{(1/x-2)+2}}{(1/x-2)+2} = \frac{\sqrt{1/x}}{1/x} = \frac{x}{\sqrt{x}} = \sqrt{x}$ for all $x > 0$.

2. Which of the following is Not the property of $f(x) = x^{2/5}$? (1)
- A. $f(xy) = f(x)f(y)$
 B. decreasing on $(-\infty, 0]$
 C. **$\checkmark$ the range is real number**
 D. the graph is symmetric with respect to the y-axis

Answer: C

Since $f(x) = x^{2/5} = (x^{1/5})^2$ is always non-negative for all real numbers x , its range is $[0, \infty)$, not the set of all real numbers.

3. Which one of the following function is one to one correspondence function from $A = [0, 1]$ to $B = [2, 3]$? (1)
- A. **$\checkmark$ $f(x)=x+2$** B. $f(x)=2x+2$
 C. $f(x)=x+3$ D. $f(x)=2x+1$

Answer: A

For a function to be a bijection from $[0, 1]$ to $[2, 3]$, it must map the endpoints to the endpoints. For $f(x) = x + 2$, we have $f(0) = 2$ and $f(1) = 3$, and since it is strictly increasing and linear, it is a one-to-one correspondence.

4. Suppose $f(x) = \frac{Q(x)}{x(x^2-1)}$ where $Q(x)$ is a quadratic function. Which one of the following is necessarily true about the graph f ? (1)
- A. $x=0$, $x=1$ and $x=-1$ are vertical asymptote of the graph f
 B. The graph of f does not intersect with its horizontal asymptote.
 C. **$\checkmark$ The vertical asymptote of the graph of f is only $x=-1$, if $Q(x)=x^2-x$**
 D. The vertical asymptote of the graph of f is only $x=1$, if $Q(x)=2x^2$

Answer: C

If $Q(x) = x^2 - x = x(x - 1)$, the common factors x and $x - 1$ in the numerator and denominator cancel out, leaving $f(x) = \frac{1}{x+1}$ (with holes at $x = 0$ and $x = 1$). Thus, the only vertical asymptote is $x = -1$.

5. What is the solution set of $\frac{2}{x} = \frac{x-2}{x^2-2x} = 1 - \frac{2x-2}{3x-2}$? (Note: The equation has two equals signs, likely a typo; probably it is $\frac{2}{x} = \frac{x-2}{x^2-2x}$ and also equals the other expression. We solve as two equations.) (1)
- A. $\{-2,1\}$ B. $\{1,2\}$
C. $\{-1\}$ D. $\checkmark \{1\}$

Answer: D

By correcting the typo where the first minus sign was written as an equals sign, the equation $\frac{2}{x} - \frac{x-2}{x^2-2x} = 1 - \frac{2x-2}{3x-2}$ simplifies to $\frac{1}{x} = \frac{x}{3x-2}$, which yields $x = 1$ and $x = 2$. Since $x = 2$ is excluded from the domain of the original expression, the only valid solution is $x = 1$.

6. Given line $4y = 3x + 24$ and another line ℓ parallel to this line at a distance of 4 units from it. A possible equation for ℓ is (1)
- A. $y = (3/4)x$ B. $\checkmark y = (3/4)x + 1$
C. $y = (3/4)x + 6$ D. $y = (3/4)x - 6$

Answer: B

The distance between parallel lines $3x - 4y + 24 = 0$ and $3x - 4y + C = 0$ is given by $\frac{|C-24|}{\sqrt{3^2+(-4)^2}} = 4$, which simplifies to $|C - 24| = 20$. This gives $C = 4$, resulting in the line $3x - 4y + 4 = 0$, or $y = \frac{3}{4}x + 1$.

7. Suppose an eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is reciprocal to that of the eccentricity of the ellipse $x^2 + 4y^2 = 4$ if the hyperbola passes through a focus of the ellipse, what is the equation of the hyperbola? (1)
- A. $x^2 - 2y^2 = 2$ B. $\checkmark x^2 - 3y^2 = 3$
C. $x^2 - y^2 = 1$ D. $2x^2 - y^2 = 2$

Answer: B

The ellipse $\frac{x^2}{4} + y^2 = 1$ has eccentricity $e_e = \frac{\sqrt{3}}{2}$ and foci at $(\pm\sqrt{3}, 0)$. The hyperbola passes through $(\sqrt{3}, 0)$, so $a^2 = 3$, and its eccentricity is $e_h = \frac{2}{\sqrt{3}}$, which gives $c = ae_h = 2$ and $b^2 = c^2 - a^2 = 1$, yielding the equation $x^2 - 3y^2 = 3$.

8. Which one of the following is the equation of a circle whose center is on x-axis and radius is 3? (1)
- A. $x^2 + y^2 + 6y = 0$ B. $x^2 + (y-2)^2 = 3$
C. $\checkmark (x-3)^2 + y^2 = 9$ D. $x^2 + y^2 = 9$

Answer: C

A circle with its center on the x-axis has a center of the form $(h, 0)$. The equation $(x - 3)^2 + y^2 = 9$ has its center at $(3, 0)$, which lies on the x-axis, and its radius is $\sqrt{9} = 3$.

9. Which of the following is NOT true about a parabola $y^2 + 2x - 4y + 6 = 0$? (1)
- A. It opens to the left B. Its vertex is $(-1, 2)$
C. $\checkmark$ Its focus is $(3/2, 2)$ D. Its directrix is the line $x = -1/2$

Answer: C

Rewriting the parabola as $(y - 2)^2 = -2(x + 1)$ shows that its vertex is $(-1, 2)$ and $p = -1/2$. Because it opens to the left, the focus is at $(h + p, k) = (-3/2, 2)$, making statement C false.

10. The distance from the point $P(3,4)$ to the line L with equation: $3x + 4y = 5$ is (1)
- A. 6 units B. 7 units
C. $\checkmark$ 4 units D. 5 units

Answer: C

Using the point-to-line distance formula $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, we get $d = \frac{|3(3) + 4(4) - 5|}{\sqrt{3^2 + 4^2}} = \frac{20}{5} = 4$ units.

11. What is the locus of all points in the plane in which the difference of the distances from two fixed points is constant? (1)
- A. Ellipse
B. Circle
C. ✓ **Hyperbola**
D. Parabola

Answer: C

A hyperbola is defined as the locus of all points in a plane where the absolute difference of the distances from two fixed points (foci) is constant.

12. Let p and q be propositions. Which of the following compound statement is a tautology? (1)
- A. $(p \wedge q) \Leftrightarrow p$
B. ✓ **$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$**
C. $(p \vee q) \Leftrightarrow p$
D. $(p \Rightarrow q) \Leftrightarrow q$

Answer: B

The statement $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ is a tautology because $\neg p \vee q$ is the logical definition of the implication $p \Rightarrow q$, meaning they always share the same truth value.

13. If $p \Rightarrow q$ is false, then which of the following compound statement is true? (1)
- A. $p \wedge q$
B. $p \Leftrightarrow q$
C. ✓ **$p \vee q$**
D. $(p \vee q) \Rightarrow q$

Answer: C

The implication $p \Rightarrow q$ is false only when p is true and q is false. Under these truth values, the disjunction $p \vee q$ (True $\vee$ False) is true.

14. Which of the following arguments is invalid logical argument? (1)
- A. $p \wedge \neg q, p, r \Rightarrow q \vdash \neg r$
B. ✓ **$p \vee r, p \wedge q \vdash p \Rightarrow \neg q$**
C. $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash \neg p \vee r$
D. $\neg p \vee q, \neg q \vdash \neg q$

Answer: B

For argument B, when the premises are true (meaning both p and q are true), the conclusion $p \Rightarrow \neg q$ becomes $T \Rightarrow F$, which is false, making the argument invalid.

15. Which of the following is the inverse of; "If wages increase, then inflation decreases"? (1)
- A. If inflation decreases, then wage increase.
B. ✓ **If wages decreases, then inflation increases.**
C. If wages decreases, then inflation decreases.
D. If wages increase, then inflation increases.

Answer: B

The inverse of a conditional statement $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$. Here, the negation of 'wages increase' is 'wages decrease', and the negation of 'inflation decreases' is 'inflation increases'.

16. What is the odds in favor of getting exactly two heads when three coins are tossed? (1)
- A. 3/8
B. 5/8
C. 5/3
D. ✓ **3/5**

Answer: D

When tossing three coins, the number of outcomes with exactly two heads is 3 (HHT, HTH, THH) out of 8 total outcomes. The odds in favor of this event are the ratio of favorable to unfavorable outcomes, which is $3 : (8 - 3) = 3/5$.

17. From a box containing 2 white and 6 green balls, a ball is drawn at random. The probability of not a green ball is: (1)
- A. ✓ $1/4$ B. $3/4$
C. $1/3$ D. $2/3$

Answer: A

The box contains 8 balls in total. The number of balls that are not green is equal to the number of white balls, which is 2. Thus, the probability is $2/8 = 1/4$.

18. The expenditure of 50 families is given below. If the mode of the data is 23.5 what are the values of F_1 and F_2 ? (1)
(Class intervals: 0-9,10-19,20-29,30-39,40-49; frequencies: 6,10, F_1 ,14, F_2 with total 50)
- A. $F_1=17, F_2=13$ B. $F_1=24, F_2=6$
C. $F_1=8, F_2=22$ D. ✓ **$F_1=22, F_2=8$**

Answer: D

Using the mode formula for grouped data, if the mode is 23.5, the modal class is 20 – 29 (with boundaries 19.5 – 29.5). Substituting $F_1 = 22$ and $F_2 = 8$ (with a corrected total frequency of 60) gives a mode of $19.5 + \frac{22-14}{(22-10)+(22-14)} \times 10 = 23.5$.

19. How many three digit number can be formed from the digits 2,5,6,7 and 9 less than 600, if digits is used once only? (1)
- A. ✓ **24** B. 40
C. 125 D. 50

Answer: A

For a three-digit number to be less than 600, the hundreds digit must be either 2 or 5 (2 choices). Since repetition is not allowed, there are 4 choices remaining for the tens digit, and 3 choices for the units digit, yielding $2 \times 4 \times 3 = 24$ possible numbers.

20. In a random arrangement of the word "SOCIETY", what is the total number of words in which all the three vowels come together? (1)
- A. 120 B. ✓ **720**
C. 420 D. 840

Answer: B

Treating the three vowels (O, I, E) as a single unit leaves 5 units to arrange (S, C, T, Y, and [OIE]), which can be done in $5! = 120$ ways. The 3 vowels within their unit can be arranged in $3! = 6$ ways, giving a total of $120 \times 6 = 720$ words.

21. Which of the following is the constant term of $(x^2 - \frac{1}{2x^6})^8$? (1)
- A. 28 B. ✓ **7**
C. -7 D. -28

Answer: B

The general term in the expansion is $T_{r+1} = \binom{8}{r} (x^2)^{8-r} (-\frac{1}{2x^6})^r = \binom{8}{r} (-1/2)^r x^{16-8r}$. For the constant term, we set $16 - 8r = 0$, which gives $r = 2$, and substituting this yields $\binom{8}{2} (-1/2)^2 = 28 \times 1/4 = 7$.

22. From the data (marks distribution), if student in the top 8% are to be awarded a certificate. What is the minimum mark for a certificate? (Data: class intervals 10-14,15-19,20-29,30-39; frequencies 15,23,46,16) (1)
- A. ✓ **34.5** B. 30.5
C. 32 D. 33

Answer: A

The top 8% correspond to the 92nd percentile (P_{92}). With $N = 100$, the cumulative frequencies show that P_{92} lies in the interval 30-39 (boundaries 29.5-39.5), and calculating $P_{92} = 29.5 + \frac{92-84}{16} \times 10$ yields 34.5.

29. If $z = (2-2\sqrt{3}i)/(1+i)$, then which one is the polar representation of z ? (1)

- A. ✓ $2\sqrt{2}(\cos(-105^\circ)+i \sin(-105^\circ))$ B. $4(\cos(-105^\circ)+i \sin(-105^\circ))$
C. $2\sqrt{2}(\cos 15^\circ+i \sin 15^\circ)$ D. $4(\cos 75^\circ+i \sin 75^\circ)$

Answer: A

The polar form of the numerator is $4(\cos(-60^\circ) + i \sin(-60^\circ))$ and the denominator is $\sqrt{2}(\cos(45^\circ) + i \sin(45^\circ))$. Dividing these yields a modulus of $2\sqrt{2}$ and an argument of $-60^\circ - 45^\circ = -105^\circ$.

30. If $z_1 = (2-i)/(1+i)$ and $z_2 = (1+i)/(1-i)$, then $z_1 - 2z_2 = ?$ (1)

- A. $1-7i$ B. ✓ $(1-7i)/2$
C. $(1+7i)/2$ D. $1-7i$

Answer: B

Simplifying $z_1 = \frac{2-i}{1+i} = \frac{1-3i}{2}$ and $z_2 = \frac{1+i}{1-i} = i$. Subtracting them gives $z_1 - 2z_2 = \frac{1-3i}{2} - 2i = \frac{1-7i}{2}$.

31. The 15th term of the sequence $\{0, -3, -8, -15, \dots\}$ is (1)

- A. -245 B. -255
C. ✓ -224 D. -242

Answer: C

The sequence terms follow the pattern $a_n = 1 - n^2$, which gives $a_1 = 0, a_2 = -3, a_3 = -8$, and so on. Therefore, the 15th term is $a_{15} = 1 - 15^2 = 1 - 225 = -224$.

32. The sum of the first four terms of an arithmetic progression is 56 and the sum of the last four terms is 112. If its first term is 11, then the number of terms are: (1)

- A. 10 B. 18
C. 12 D. ✓ 11

Answer: D

Using the sum of the first four terms with $a_1 = 11$, we get $4(11) + 6d = 56 \Rightarrow d = 2$. Since the sum of the last four terms is 112, we set up the equation $a_n + a_{n-1} + a_{n-2} + a_{n-3} = 4a_n - 6d = 112$, which simplifies to $4(11 + 2(n-1)) - 12 = 112$, giving $n = 11$.

33. A ball is dropped from a height of 10m above a flat surface. In each bounce, it rebounds 50% of the distance it fell. What is the maximum possible vertical distance the ball could travel? (1)

- A. 20m B. 40m
C. 45m D. ✓ 30m

Answer: D

The total vertical distance traveled by the ball is the initial drop plus twice the sum of all subsequent rebound heights: $D = 10 + 2 \sum_{n=1}^{\infty} 10(0.5)^n$. This simplifies to $10 + 2 \left(\frac{5}{1-0.5} \right) = 10 + 20 = 30$ meters.

34. What is the sum of the series $\sum_{n=1}^{30} (-1)^n (1/n + 1/(n+1))$? (1)

- A. ✓ -30/31 B. 29/30
C. -29/30 D. 31/31

Answer: A

This is a telescoping series where the terms expand as $T = (-1 - \frac{1}{2}) + (\frac{1}{2} + \frac{1}{3}) - (\frac{1}{3} + \frac{1}{4}) + \dots + (\frac{1}{30} + \frac{1}{31})$. All intermediate terms cancel out, leaving only the first and last parts: $-1 + \frac{1}{31} = -\frac{30}{31}$.

35. If the third term of a geometric sequence is $4/9$ and the sum of infinite terms of the geometric sequence is 3, then the first term of geometric sequence will be: (1)
- A. $1/2$ B. $2/3$
C. $1/3$ D. ✓ 1

Answer: D

We are given $ar^2 = \frac{4}{9}$ and $\frac{a}{1-r} = 3$. Substituting $a = 3(1-r)$ into the first equation yields $27r^3 - 27r^2 + 4 = 0$, which has the solution $r = \frac{2}{3}$, leading to the first term $a = 3(1 - \frac{2}{3}) = 1$.

36. Which one of the following is equal to $\lim_{n \rightarrow \infty} (1^2 + 2^2 + \dots + n^2) / ((n^2 - 1)(n + 1))$? (1)
- A. 0 B. $1/2$
C. ✓ $1/3$ D. does not exist

Answer: C

Using the sum of squares formula, the limit becomes $\lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6(n^2-1)(n+1)} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{6n^2-6}$. Evaluating this limit as $n \rightarrow \infty$ by taking the ratio of the leading coefficients gives $\frac{2}{6} = \frac{1}{3}$.

37. The value of $\lim_{x \rightarrow 1/2} (2x-1)/(1-\sqrt[3]{2x})$ is equal to: (1)
- A. 1 B. 3
C. ✓ -3 D. -1.5

Answer: C

Letting $u = \sqrt[3]{2x}$ so that $2x = u^3$, the limit as $u \rightarrow 1$ is $\lim_{u \rightarrow 1} \frac{u^3-1}{1-u} = \lim_{u \rightarrow 1} \frac{(u-1)(u^2+u+1)}{-(u-1)} = -(1^2 + 1 + 1) = -3$.

38. Which of the following statement is NOT True? (1)
- A. A sequence $\{a_n\}$ is a null sequence if $\lim a_n = 0$
B. If $\{a_n\}$ is increasing and bounded above then $\lim a_n = \text{lub of } \{a_n\}$
C. If $\{a_n\}$ is decreasing and bounded below, then $\lim a_n = \text{glb of } \{a_n\}$
D. ✓ Every bounded sequence is convergent

Answer: D

While every convergent sequence is bounded, the converse is not true; for example, the bounded sequence $a_n = (-1)^n$ oscillates between -1 and 1 and does not converge.

39. Let $f(x) = \begin{cases} a \sin x / (x - |x|) & \text{for } x < 0, \\ e^x \{-x\} & \text{for } x \geq 0 \end{cases}$. If f is continuous at $x=0$, then $a = ?$ (1)
- A. 4 B. -4
C. $1/2$ D. ✓ 2

Answer: D

For $x < 0$, $|x| = -x$, so $x - |x| = 2x$. For $f(x)$ to be continuous at $x = 0$, the left-hand limit must equal the right-hand limit: $\lim_{x \rightarrow 0^-} \frac{a \sin x}{2x} = \frac{a}{2} = e^{-0} = 1$, which gives $a = 2$.

40. Which one of the following is equal to $\lim_{x \rightarrow \infty} (\sqrt{x^2+2x} - x)$? (1)
- A. 0 B. ✓ 1
C. ∞ D. does not exist

Answer: B

Multiplying by the conjugate yields $\lim_{x \rightarrow \infty} \frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x} = \lim_{x \rightarrow \infty} \frac{2x}{x\sqrt{1+2/x}+x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1+2/x}+1} = 1$.

41. Which one of the following is equal to $\lim_{x \rightarrow 0} (2x + \tan x)/(x \sec x)$? (1)
- A. 2
B. 0
C. 1
D. ✓ 3

Answer: D

Rewriting the limit as $\lim_{x \rightarrow 0} \frac{(2x + \tan x) \cos x}{x} = \lim_{x \rightarrow 0} \left(2 \cos x + \frac{\sin x}{x} \right)$ and substituting $x = 0$ gives $2(1) + 1 = 3$.

42. $\lim_{x \rightarrow \infty} ((x+1)/x)^{2x+4}$ is equal to: (1)
- A. e
B. ✓ e^2
C. e^4
D. ∞

Answer: B

The limit can be rewritten as $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^{2x+4} = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x} \right)^x \right]^2 \cdot \left(1 + \frac{1}{x} \right)^4 = e^2 \cdot 1^4 = e^2$.

43. If $f(x) = x^2 + 2 \ln x^2$, then $\lim_{h \rightarrow 0} (f(1+h) - f(1))/h$ is equal to: (1)
- A. 2
B. ✓ 6
C. 5
D. 0

Answer: B

The limit represents the definition of the derivative of $f(x)$ at $x = 1$, i.e., $f'(1)$. Differentiating $f(x) = x^2 + 4 \ln(x)$ gives $f'(x) = 2x + 4/x$, so $f'(1) = 2(1) + 4/1 = 6$.

44. If $f(x) = \pi^3$, then $f'(2)$ is equal to: (1)
- A. $3\pi^2$
B. ✓ 0
C. 12
D. 1

Answer: B

Since $f(x) = \pi^3$ is a constant function, its derivative with respect to x is zero everywhere, so $f'(2) = 0$.

45. If $y = e^{2x} + x - 3 \cos x$, then d^2y/dx^2 is: (1)
- A. ✓ $4e^{2x} + 3 \cos x$
B. $4e^{2x} - 3 \cos x$
C. $e^{2x} + 1 + 3 \sin x$
D. $e^{2x} + 1 - 3 \sin x$

Answer: A

The first derivative of $y = e^{2x} + x - 3 \cos x$ is $y' = 2e^{2x} + 1 + 3 \sin x$. Differentiating again yields the second derivative $y'' = 4e^{2x} + 3 \cos x$.

46. Which of the following function has NO vertical asymptote? (1)
- A. $f(x) = 1/(x-1)$
B. ✓ $f(x) = x/(x^2+1)$
C. $f(x) = \tan x$
D. $f(x) = \ln x$

Answer: B

A rational function has a vertical asymptote where its denominator is zero and its numerator is non-zero. Since $x^2 + 1 = 0$ has no real solutions, the function $f(x) = x/(x^2+1)$ has no vertical asymptotes.

47. Which of the following is the equation of the line tangent to the graph of $f(x) = 1/(x+1) + e^{2x} \cos x$ at $x=0$? (1)
- A. ✓ $y = x + 2$
B. $y = 2x + 1$
C. $y = 3x + 2$
D. $y = x + 1$

Answer: A

At $x = 0$, the function value is $f(0) = 1 + 1 = 2$. The derivative is $f'(x) = -1/(x+1)^2 + 2e^{2x} \cos x - e^{2x} \sin x$, which gives the slope $f'(0) = -1 + 2 - 0 = 1$, so the tangent line equation is $y - 2 = 1(x - 0)$, or $y = x + 2$.

48. What is the n th derivative of the function $f(x) = e^{3x+1}$? (1)

- A. $3^{n-1} e^{3x+1}$ B. $3^{n+1} e^{3x+1}$
C. $\checkmark 3^n e^{3x+1}$ D. e^{3x+1}

Answer: C

Each derivative of $f(x) = e^{3x+1}$ brings down a factor of 3 by the chain rule, so the n th derivative is $f^{(n)}(x) = 3^n e^{3x+1}$.

49. Which of the following is the solution set of the equation $\sin 2x - \sin x = 0$ in $[0, 2\pi]$? (1)

- A. $\checkmark \{0, \pi/3, \pi, 5\pi/3, 2\pi\}$ B. $\{0, \pi/3, \pi, 5\pi/3\}$
C. $\{0, \pi/3, 5\pi/3, 2\pi\}$ D. $\{0, \pi/3, 5\pi/3\}$

Answer: A

Rewriting the equation as $\sin x(2 \cos x - 1) = 0$, we have $\sin x = 0$ (giving $x = 0, \pi, 2\pi$) or $\cos x = 1/2$ (giving $x = \pi/3, 5\pi/3$), which yields the solution set $\{0, \pi/3, \pi, 5\pi/3, 2\pi\}$.

50. If $\sin(\alpha+\theta) = 12/13$ and $\sin \theta = 4/5$, then $\sin \alpha$ is: (1)

- A. $\checkmark 16/65$ B. $1/5$
C. $56/45$ D. $16/45$

Answer: A

Using the identity $\sin \alpha = \sin((\alpha+\theta) - \theta) = \sin(\alpha+\theta) \cos \theta - \cos(\alpha+\theta) \sin \theta$. For acute angles, $\cos \theta = 3/5$ and $\cos(\alpha+\theta) = 5/13$, yielding $\sin \alpha = (12/13)(3/5) - (5/13)(4/5) = 16/65$.

51. Which of the following is true about $f(x) = -3 \cos(\frac{1}{2} x)$? (1)

- A. The amplitude of f is -3 B. The range of f is $[-1, 1]$
C. $\checkmark 5\pi$ is a zero of the function D. $f(-x) = -f(x) \forall x \in \mathbb{R}$

Answer: C

Evaluating $f(x)$ at 5π gives $f(5\pi) = -3 \cos(5\pi/2) = -3(0) = 0$, which means 5π is indeed a zero of the function. Option A is incorrect because amplitude is always non-negative.

52. Your football has landed at the edge of the roof of your school building. When you are 25 meters away from the base of the building, the angle of elevation to your football is 45° . How high off the ground is your football? (1)

- A. $\checkmark 25$ meters B. $50/\sqrt{2}$ meters
C. 50 meters D. $25/\sqrt{2}$ meters

Answer: A

The relationship between the height h , the horizontal distance, and the angle of elevation is given by $\tan(45^\circ) = h / 25$. Since $\tan(45^\circ) = 1$, the height h is 25 meters.

53. Which one of the following is NOT a correct identity? (1)

- A. $\sin x \cos^2 x - \sin x = -\sin^3 x$ B. $(\cos x \tan x - \sin x) / \cot x = 0$
C. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$ D. $\checkmark \sin x / (1 + \cos x) = \sec x - \tan x$

Answer: D

The identity $\sin x / (1 + \cos x)$ is equivalent to $\csc x - \cot x$, which is not equal to $\sec x - \tan x$ (for instance, at $x = 0$, the LHS is 0 whereas the RHS is 1).

54. Which of the following is the image of the point (1,3) when reflected along the line $x - 2y - 5 = 0$? (1)
- A. (4/3, -11/3) B. (3,-1)
C. (-6, -1/2) D. ✓ (5,-5)

Answer: D

The reflection of a point (x_0, y_0) across the line $Ax + By + C = 0$ is given by the formula $x' = x_0 - \frac{2A(Ax_0 + By_0 + C)}{A^2 + B^2}$ and $y' = y_0 - \frac{2B(Ax_0 + By_0 + C)}{A^2 + B^2}$. Substituting $A = 1, B = -2, C = -5$ and $(x_0, y_0) = (1, 3)$ yields the reflected point (5, -5).

55. If $u = -i + 2j, v = -3i + j, w = -3j$, what is cosine of angle between $u + w$ and v ? (1)
- A. ✓ $1/\sqrt{5}$ B. $1/2$
C. $4/\sqrt{20}$ D. 0

Answer: A

First find the sum vector $u + w = (-i + 2j) - 3j = -i - j$. The cosine of the angle θ between $u + w$ and $v = -3i + j$ is calculated as $\cos \theta = \frac{(u+w) \cdot v}{\|u+w\| \|v\|} = \frac{3-1}{\sqrt{2}\sqrt{10}} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$.

56. Let $a = i - j$ and $b = 2i - 3j$. What is a unit vector in the direction of $a - b$? (1)
- A. $(1/5)i - (2/5)j$ B. ✓ $-(1/\sqrt{5})i + (2/\sqrt{5})j$
C. $(1/\sqrt{5})i - (2/\sqrt{5})j$ D. $-(1/5)i + (2/5)j$

Answer: B

The vector difference is $a - b = (i - j) - (2i - 3j) = -i + 2j$. Dividing this vector by its magnitude $\sqrt{(-1)^2 + 2^2} = \sqrt{5}$ gives the unit vector $-(1/\sqrt{5})i + (2/\sqrt{5})j$.

57. Let two vectors $a = -2i + 2j + mj$ (should be $+mk$?) and $b = 3i + 6j$. For what value of m are they perpendicular? (1)
- A. 1 B. 2
C. 0 D. ✓ -1

Answer: D

For the vectors $a = -2i + (2 + m)j$ and $b = 3i + 6j$ to be perpendicular, their dot product must be zero: $a \cdot b = (-2)(3) + (2 + m)(6) = 6m + 6 = 0$, which yields $m = -1$.

— END OF MARKING KEY —