

Mathematics, 2007 EC, entrance exam

Grade 12 · Mathematics · 2007 E.C.

MARKING KEY — CONFIDENTIAL

Time allowed: 120 minutes

Questions: 62

Total marks: 62

1. Which one of the following is the equation of an ellipse with centre at (1, 4), vertices at (10, 4) and (1, 2) is : (1)
- A. $\checkmark$ $4(x - 1)^2 + 81(y - 4)^2 = 324$ B. $(x - 1)^2 + 9(y - 4)^2 = 4$
 C. $9(x - 1)^2 + 4(y - 4)^2 = 1$ D. $2(x - 1)^2 + 9(y - 4)^2 = 4$

Answer: A

The horizontal semi-axis is $a = |10 - 1| = 9$ and the vertical semi-axis is $b = |4 - 2| = 2$. The standard equation is $\frac{(x-1)^2}{81} + \frac{(y-4)^2}{4} = 1$, which simplifies to $4(x - 1)^2 + 81(y - 4)^2 = 324$.

2. Which of the following is convergent sequence? (1)
- A. $1, 1/2, 1, 1/3, 1, 1/4, \dots$ B. $\{(-1)^n\}_{n=1}^{\infty}$
 C. $\checkmark$ $\{10^{109} - \frac{1}{100^n}\}_{n=1}^{\infty}$ D. $\{n + \sin \frac{1}{n}\}_{n=1}^{\infty}$

Answer: C

As $n \rightarrow \infty$, $\frac{1}{100^n} \rightarrow 0$, so the sequence converges to the constant 10^{109} . The other sequences diverge by oscillation or tending to infinity.

3. Let $f(x) = 2x(x^2 + 1)^4$, then which of the following is an anti- derivative of f(x)? (1)
- A. $\frac{2x}{5}(x^2 + 1)^5 + c$ B. $\frac{2}{5}(x^2 + 1)^5 + c$
 C. $\frac{x}{5}(x^2 + 1)^5 + 1$ D. $\checkmark$ $\frac{1}{5}(x^2 + 1)^5 - 1$

Answer: D

Using the substitution $u = x^2 + 1$ with $du = 2x dx$, we find $\int 2x(x^2 + 1)^4 dx = \frac{1}{5}(x^2 + 1)^5 + C$. Setting the constant of integration $C = -1$ gives the anti-derivative in option D.

4. Which one of the following is an arithmetic sequence? (1)
- A. $\checkmark$ **3, 5, 7, 9, 11, ...** B. 3, 6, 12, 24, 28, ...
 C. -3, 6, -9, 12, -15, ... D. 1, 3, 6, 10, 15, 21, ...

Answer: A

An arithmetic sequence has a constant difference between consecutive terms; for 3, 5, 7, 9, 11, the common difference is $d = 2$.

5. Suppose that p represents the statement "He missed the tournament", q represents the statement "He got the gold medal." And r represents the statement "he took a trip abroad." then which of the following symbolic expressions represents the statement: "If He took a trip abroad and he does not miss the tournament, then he get the gold medal."? (1)
- A. $(r \Rightarrow q) \neg p$ B. $r \wedge (p \Rightarrow q)$
 C. $\checkmark$ $(r \wedge \neg p \Rightarrow q)$ D. $\neg(r \vee p) \vee q$

Answer: C

'He took a trip abroad' is r , 'he does not miss the tournament' is $\neg p$, and their conjunction implies 'he got the gold medal' (q), which is symbolically written as $(r \wedge \neg p \Rightarrow q)$.

6. Let $f(x) = \begin{cases} a \frac{\sin x}{x}, & x < 0 \\ e^{2x} - 2, & x \geq 0 \end{cases}$ if f is continuous at $x=0$, then what is the value of a ? (1)
- A. $1/2$ B. 2
C. $\checkmark -1/2$ D. -2

Answer: C

7. The Ozone level (in ppb- parts per billion) on a sunny day in metropolitan area is given by formula $p(t) = 80 + 12t - t^2$ where t is time in hours and $t = 0$ corresponds to 9A.M. what is the rate of increase of Ozone level after 3-hrs(i.e. at 12 A.M)? (1)
- A. $\checkmark 6$ ppb B. 12 ppb
C. 107 ppb D. 113 ppb

Answer: A

The rate of increase is given by the derivative $p'(t) = 12 - 2t$. At $t = 3$ hours, $p'(3) = 12 - 2(3) = 6$ ppb.

8. Which of the following functions is one to one correspondence? (1)
- A. $f: R \rightarrow R, f(x) = \tan x$, where R' is domain of f B. $g: R \rightarrow R, g(x) = 2^x$
C. $h: [-\infty, \infty) \rightarrow [0, \infty): h(x) = x^2$ D. $\checkmark r: [0, \infty) \rightarrow [0, \infty): r(x) = x + 5$

Answer: D

9. Different codes, each of which consisting of five characters, are to be generated in such a way that the first two characters are any of the English capital letters (A to Z) and the remaining three are any of the digits (0, 1, 2, ..., 9). How many distinct codes can be generated? (1)
- A. 468,000 B. 260
C. $\checkmark 676,000$ D. $26! \times 10!$

Answer: C

There are 26 choices for each of the first two letters and 10 choices for each of the three digits, yielding $26 \times 26 \times 10 \times 10 \times 10 = 676,000$ distinct codes.

10. A certain meeting hall has 20 rows of seats. There are 20 seats in the first row, 22 seats in the second row, 24 seats in the third row, and so on. How many seats are there on the last (20th) row of the hall? (1)
- A. 46 B. $\checkmark 58$
C. 76 D. 5240

Answer: B

This is an arithmetic sequence with first term $a_1 = 20$ and common difference $d = 2$. The 20th term is $a_{20} = a_1 + 19d = 20 + 19(2) = 58$.

11. Two perpendicular lines l_1 and l_2 are intersecting at $(-1, 2)$. If the angle of inclination of l_1 is 45° , then what is the equation of l_2 ? (1)
- A. $y = -x + 3$ B. $y = x + 3$
C. $\checkmark y = -x + 1$ D. $y = x + 1$

Answer: C

The slope of l_1 is $m_1 = \tan(45^\circ) = 1$, so the perpendicular line l_2 has slope $m_2 = -1$. Using the point-slope form with $(-1, 2)$, the equation of l_2 is $y - 2 = -1(x + 1)$, which simplifies to $y = -x + 1$.

12. What is the area of the region between the graph of $y = \sin x$ and x -axis where $0 \leq x \leq 2\pi$? (1)

- A. ✓ 4
B. 4π
C. 2
D. 2π

Answer: A

The area between $y = \sin x$ and the x -axis on $[0, 2\pi]$ is given by $\int_0^\pi \sin x \, dx + \int_\pi^{2\pi} -\sin x \, dx = 2 + 2 = 4$.

13. The following is the set of data representing the average mark of students: 91, 89, 93, 91, 87, 94, 92, 85, 91, 90, 96, 93, 89. Then which one of the following statements is true about the data? (1)

- A. the median is 90.5
B. the upper quartile is 92
C. ✓ the range of marks is 11
D. the mean is 91.5

Answer: C

Arranging the 13 numbers in ascending order gives a minimum of 85 and a maximum of 96. The range is $96 - 85 = 11$.

14. If $f(x) = \ln(\sqrt{x^2 + 1})$, which of the following is equal to $f'(x)$? (1)

- A. $\frac{x}{\sqrt{x^2 + 1}}$
B. ✓ $\frac{x}{x^2 + 1}$
C. $\frac{2x}{\sqrt{x^2 + 1}}$
D. $\frac{2x}{x^2 + 1}$

Answer: B

Rewriting $f(x) = \frac{1}{2} \ln(x^2 + 1)$, we differentiate using the chain rule to obtain $f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2 + 1} = \frac{x}{x^2 + 1}$.

15. If $z = \frac{3+i}{i-2}$ is a given complex number, then what is the conjugate $\bar{z}$ of z ? (1)

- A. ✓ $\bar{z} = \frac{-3+i}{2+i}$
B. $\bar{z} = \frac{i-3}{2-i}$
C. $\bar{z} = -6 - 2i$
D. $\bar{z} = -1 - i$

Answer: A

Simplifying gives $z = \frac{3+i}{-2+i} = -1 - i$, so its conjugate is $\bar{z} = -1 + i$. Taking the conjugate of the numerator and denominator directly gives $\bar{z} = \frac{3-i}{-i-2} = \frac{-3+i}{2+i}$.

16. What is the principal argument of $(5 + 5i)^{11}$? (1)

- A. $\pi/2$
B. $2\pi/3$
C. $\pi/4$
D. ✓ $3\pi/4$

Answer: D

The argument of $5 + 5i$ is $\pi/4$. Therefore, the argument of $(5 + 5i)^{11}$ is $11 \times \frac{\pi}{4} = \frac{11\pi}{4} = 2\pi + \frac{3\pi}{4}$, giving a principal argument of $\frac{3\pi}{4}$.

17. Which one of the following is equal to $\lim_{x \rightarrow 9} \frac{x-9}{3-\sqrt{x}}$? (1)

- A. 6
B. -3
C. ✓ -6
D. ∞

Answer: C

Factoring the numerator gives $x - 9 = -(\sqrt{x} - 3)(\sqrt{x} + 3) = -(3 - \sqrt{x})(3 + \sqrt{x})$. Canceling $(3 - \sqrt{x})$ leaves $\lim_{x \rightarrow 9} -(3 + \sqrt{x}) = -(3 + 3) = -6$.

18. For arbitrary propositions p and q , which one of the following is a valid equivalence? (1)

- A. $\neg(p \Rightarrow q) \equiv (q \Rightarrow p)$ B. $\checkmark \quad [\neg(p \Rightarrow q) \wedge p] \equiv (p \wedge \neg q)$
C. $[p \vee \neg q] \equiv p \Rightarrow q$ D. $[(p \vee q) \Rightarrow q] \equiv [p \Rightarrow \neg q]$

Answer: B

Since $\neg(p \Rightarrow q) \equiv p \wedge \neg q$, the conjunction $[\neg(p \Rightarrow q) \wedge p] \equiv (p \wedge \neg q) \wedge p \equiv p \wedge \neg q$.

19. If f and g are continuous on $\mathbb{R}$ and $a, b \in \mathbb{R}$, which one of the following is necessarily true? (1)

- A. if $\int_a^b f(x)dx = \int_a^b g(x)dx$, then $f(x) = g(x)$ for all $x \in [a, b]$ B. if $f'(x) = g'(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx = \int_a^b g(x)dx$
C. if $f(x) \geq 3$ for all $x \in [-2, 2]$, then $\checkmark \quad \int_{-2}^2 f(x)dx \geq 12$ D. $\int_a^b f(x)dx = \int_b^a f(x)dx$

Answer: C

If a function satisfies $f(x) \geq m$ on $[a, b]$, then the integral from a to b of $f(x) dx$ is at least $m(b - a)$. Here, $3 * (2 - (-2)) = 3 * 4 = 12$.

20. Consider the following system of equations:
$$\begin{cases} ax + by = 2 \\ x + 3y + 2z = 0 \\ 2x + y + z = 0 \end{cases}$$
 if the determinant of the coefficient matrix is (1)

2, then what is the solution set of the system?

- A. $\checkmark \quad \{(1, 3, -5)\}$ B. $\{(1/a, 1/b, 0)\}$
C. $\{(-2, -6, 10)\}$ D. $\emptyset$

Answer: A

The determinant of the coefficient matrix gives $a + 3b = 2$. From the second and third equations, $y = 3x$ and $z = -5x$; substituting these into the first equation yields $(a + 3b)x = 2$, so $2x = 2$, giving $x = 1$, $y = 3$, and $z = -5$.

21. For what values of a and b is the function $f(x) = \begin{cases} 1 - 3x^2, & \text{for } x \leq 1 \\ ax + b, & \text{for } x > 1 \end{cases}$ is differentiable at $x=1$? (1)

- A. $a = 6, b = 0$ B. $a = -3, b = 1$
C. $a = 0, b = -2$ D. $\checkmark \quad a = -6, b = 4$

Answer: D

For differentiability at $x = 1$, the left- and right-hand derivatives must be equal, giving $a = -6(1) = -6$. For continuity at $x = 1$, $1 - 3(1)^2 = a(1) + b$, which gives $-2 = -6 + b$, so $b = 4$.

22. Which one of the following is equal to $\int \frac{\ln x + x^2 e^x}{x} dx$? (1)

- A. $\frac{1}{2} \ln^2 x + e^x(x^2 - 1) + c$ B. $\checkmark \quad \frac{1}{2} \ln^2 x + e^x(x - 1) + c$
C. $\frac{1}{x^2} \ln x + e^x(x - 1) + c$ D. $\frac{-1}{x^2} \ln x + e^x(x^2 - 1) + c$

Answer: B

Splitting the integrand gives $\int (\ln x)/x dx + \int (x e^x) dx$. The first integral is $(1/2)(\ln x)^2 + c_1$ and the second, by integration by parts, is $e^x(x - 1) + c_2$.

Multiplying both sides by 20 gives $20(u + 3i)/(4 - 2i) = (u + 3i)(4 + 2i) = (4u - 6) + (2u + 12)i = 2 + vi$. Equating real and imaginary parts gives $4u - 6 = 2$ (so $u = 2$) and $v = 2(2) + 12 = 16$.

29. What is the sum of all multiples of 3 between 20 and 200? (1)
- A. 7,227
B. ✓ 6,570
C. 6,150
D. 5,166

Answer: B

The multiples of 3 between 20 and 200 form an arithmetic progression starting at 21 and ending at 198 with common difference 3. The number of terms is $n = (198 - 21)/3 + 1 = 60$, giving a sum of $(60/2) * (21 + 198) = 30 * 219 = 6,570$.

30. The invers of the function defined by $g(x) = \frac{2x}{x+3}$ is equal to: (1)

- A. $g^{-1}(x) = -\frac{2x}{x-3}$
B. ✓ $g^{-1}(x) = -\frac{3x}{x-2}$
C. $g^{-1}(x) = -\frac{x+3}{2x}$
D. $g^{-1}(x) = -\frac{x+2}{3x}$

Answer: B

Setting $y = 2x/(x+3)$ gives $y(x+3) = 2x$, which rearranges to $x(2-y) = 3y$, or $x = 3y/(2-y) = -3y/(y-2)$. Replacing y with x yields $g^{-1}(x) = -3x/(x-2)$.

31. If $A = \begin{pmatrix} 0 & x & 0 \\ 1 & -1 & 1 \\ 0 & y & -1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, then what are the value of x and y ? (1)
- A. $x = 3, y = -2$
B. $x = 2/3, y = 1/3$
C. $x = -3, y = 2$
D. ✓ $x = 1/3, y = 2/3$

Answer: D

Using $AA^{-1} = I$, the (1,1) entry gives $0(1) + x(3) + 0(2) = 3x = 1$, so $x = 1/3$. The (3,1) entry gives $0(1) + y(3) + (-1)(2) = 3y - 2 = 0$, so $y = 2/3$.

32. If $f(2) = -e$, $f'(2) = 4$, $g(1) = -5$, $g'(1) = 1$ and $F(x) = f(2x+2)(g(x-x^2))$, then what is the value of $F'(0)$? (1)
- A. 19
B. 0
C. -20
D. ✓ -40

Answer: D

By the product and chain rules, $F'(x) = 2f'(2x+2)g(\dots) + f(2x+2)g'(\dots)(\dots)$. Evaluating at $x = 0$ gives $F'(0) = 2f'(2)g(1) + 0 = 2(4)(-5) = -40$.

33. A ladder 6m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate (speed) of 0.5 m/s, how fast the angle between the top of ladder and the wall changing when the angle is $\pi/4$ rad? (1)
- A. $\sqrt{2}/2$ rad/sec
B. ✓ $\sqrt{2}/12$ rad/sec
C. $\sqrt{2}/6$ rad/sec
D. $\sqrt{2}/3$ rad/sec

Answer: B

Let x be the distance from the wall; then $x = 6 \sin \theta$. Differentiating with respect to time gives $dx/dt = 6 \cos \theta (d\theta/dt)$. Substituting $dx/dt = 0.5$ and $\theta = \pi/4$ gives $0.5 = 6(\sqrt{2}/2)(d\theta/dt) = 3\sqrt{2}(d\theta/dt)$, so $d\theta/dt = 1/(6\sqrt{2}) = \sqrt{2}/12$ rad/sec.

For any function satisfying $f(x + y) = f(x)f(y)$, setting $x = y = 0$ gives $f(0) = [f(0)]^2$, which requires $f(0)$ to be 0 or 1; it cannot be 2, nor can it take two distinct values at 0. Therefore, no such function exists.

39. Which one of the following is true about a conic section represented by the equation $\frac{x^2}{k} + \frac{y^2}{k-9} = 1$? (1)
- A. it is a circle whose center is at origin for some $k \in \mathbb{R}$.
- B. it is an ellipse whose major axis is vertical when $k > 9$.
- C. ✓ it is a hyperbola whose foci are at $(-3, 0)$ & $(3, 0)$ when $0 < k < 9$.
- D. it is a hyperbola whose foci are at $(-3k, 0)$ & $(3k, 0)$ when $0 < k < 9$.

Answer: C

When $0 < k < 9$, the equation can be written as $\frac{x^2}{k} - \frac{y^2}{9-k} = 1$, which is a horizontal hyperbola with $a^2 = k$ and $b^2 = 9 - k$. The foci are given by $c^2 = a^2 + b^2 = k + (9 - k) = 9$, so $c = 3$, placing the foci at $(-3, 0)$ and $(3, 0)$.

40. Which of the following is NOT a tautology? (1)
- A. $[p \vee (q \Rightarrow r)] \Leftrightarrow [\neg p \Rightarrow (q \Rightarrow r)]$
- B. $p \vee (q \Rightarrow \neg q)$
- C. ✓ $p \Rightarrow [(p \Rightarrow q) \vee q]$
- D. $[p \Leftrightarrow (q \wedge \neg r)] \Leftrightarrow [\neg p \Leftrightarrow (\neg q \vee r)]$

Answer: C

41. What is the solution of the system $\begin{cases} x + y - z = 1 \\ x + 2y - 3z = 1 \\ 2x + 3y - 4z = 2 \end{cases}$? (1)
- A. $\{(0, 2, 1)\}$
- B. ✓ $\{(1 - k, 2k, k) \mid k \in \mathbb{R}\}$
- C. $\{(2k + 1, -k, k) \mid k \in \mathbb{R}\}$
- D. $\emptyset$

Answer: B

Subtracting the first equation from the second yields $y - 2z = 0$, or $y = 2z$. Substituting $y = 2z$ into the first equation gives $x + 2z - z = 1 \implies x = 1 - z$, so setting $z = k$ gives the solution set $(1 - k, 2k, k)$ for $k \in \mathbb{R}$.

42. Let $\begin{cases} 3 - 2^{2x}, & \text{if } x < 0.5 \\ \frac{2^x - 5}{x + 1}, & \text{if } x \geq 0.5 \end{cases}$, if c is a zero of f , that is, $f(c) = 0$, then which one of the following intervals must contain c ? (1)
- A. $(-\infty, 0]$
- B. $[0, 1]$
- C. $[1, 2]$
- D. ✓ $[2, 3]$

Answer: D

For $x < 0.5$, $3 - 2^{2x} > 3 - 2^1 = 1 > 0$, so there is no zero. For $x \geq 0.5$, setting $\frac{2^x - 5}{x + 1} = 0$ gives $2^x = 5$, which implies $c = \log_2 5 \approx 2.32$, an element of the interval $[2, 3]$.

43. Which of the following functions could most likely be drawn as in the Figure 1 below?

(1)

[Figure: A hand-drawn sketch of a rational function on x - and y -axes. The horizontal axis carries an arrowhead on the right and is labelled x ; the vertical axis carries an arrowhead at the top and is labelled y . A dashed vertical line to the LEFT of the y -axis is labelled " $x=-2$ " and acts as a vertical asymptote. A dashed horizontal line ABOVE the x -axis is labelled " $y=1$ " and acts as a horizontal asymptote. A single tick value " 2 " is marked on the x -axis to the right of the origin. The curve has two branches. The left branch (for x to the left of the asymptote) enters from the far left just BELOW the line $y=1$, runs almost flat, then bends downward, crosses the x -axis a little to the left of $x=-2$, and plunges steeply toward minus infinity as it approaches the dashed line $x=-2$ from the left. The right branch descends from high above (plus infinity) just to the right of the dashed line $x=-2$, drops steeply, crosses the y -axis above $y=1$, then flattens and decreases slowly toward the dashed line $y=1$ from above as x increases; this branch is drawn with a small visible break (a gap, i.e. a hole) at about $x=2$, over the marked tick " 2 ". No other points, values or shading are shown.]

- A. $\frac{x+3}{x+2}$ B. $\frac{x^2-2x}{x^2-4}$
 C. $\frac{-x^2-x+6}{x^2-4}$ D. ✓ $\frac{x^2+x-6}{x^2-4}$

Answer: D

Simplifying $\frac{x^2+x-6}{x^2-4} = \frac{(x+3)(x-2)}{(x+2)(x-2)} = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$ (for $x \neq -2$) reveals a vertical asymptote at $x = -2$, a horizontal asymptote at $y = 1$, and a hole at $x = 2$, matching the given graph.

44. A measurement is grouped in to five class intervals with the following frequency distribution. Class Intervals: 5-15 | 15-25 | 25-35 | 35-45 | 45-55 Frequency: 22 | 40 | 68 | 50 | 20 What are the first quartile Q_1 & the 75th percentile P_{75} of the measurement?

(1)

[Figure: A bordered two-row table with six columns. The first column holds the row headings. Row 1 heading "Class Intervals", then the five cells: 5-15, 15-25, 25-35, 35-45, 45-55. Row 2 heading "Frequency", then the five cells: 22, 40, 68, 50, 20.]

- A. $Q_1 = 20, P_{75} = 40$ B. $Q_1 = 22, P_{75} = 40$
 C. $Q_1 = 20, P_{75} = 39$ D. ✓ $Q_1 = 22, P_{75} = 39$

Answer: D

With $N = 200$, the first quartile Q_1 is at position 50 (in class 15-25): $Q_1 = 15 + \frac{50-22}{40} \times 10 = 22$. The 75th percentile P_{75} is at position 150 (in class 35-45): $P_{75} = 35 + \frac{150-130}{50} \times 10 = 39$.

45. The number of shoes s that a factory can produce per day is a function of the number of hours t it operates: $s(t) = 40t$ for $0 \leq t \leq 12$ the daily cost c , in Birr, to manufacture s shoes is given by the function $c(s) = 0.1s^2 + 90s + 800$ If the factory operates for 10 hours, what is the cost it incurs in producing as much shoes it can with in this time?

(1)

- A. Birr 400 B. Birr 1,600
 C. ✓ Birr 52,800 D. Birr 124,600

Answer: C

In 10 hours, the factory produces $s(10) = 40(10) = 400$ shoes. The total cost is $c(400) = 0.1(400)^2 + 90(400) + 800 = 16,000 + 36,000 + 800 = 52,800$ Birr.

46. Three persons P1, P2 and P3 are firing at a target independently and have a probability 0.7, 0.5 and 0.4, respectively, of hitting the target. What is the probability that at least one of them hits the target? (1)
- A. 0.95
B. 0.85
C. ✓ 0.91
D. 0.99

Answer: C

The probability that none of the three hits the target is $(1 - 0.7)(1 - 0.5)(1 - 0.4) = 0.3 \times 0.5 \times 0.6 = 0.09$. Thus, the probability that at least one hits is $1 - 0.09 = 0.91$.

47. Which one of the following is equal to $\lim_{x \rightarrow \infty} \left(\frac{3x}{3x+2} \right)^{-3x}$? (1)
- A. ✓ e^2
B. e^{-3}
C. e^{-2}
D. e^3

Answer: A

Inverting the base and negating the exponent gives $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{3x} \right)^{3x}$. Substituting $u = 3x$, the limit becomes $\lim_{u \rightarrow \infty} \left(1 + \frac{2}{u} \right)^u = e^2$.

48. A company manufactures x computer sets per month. The monthly marginal profit (in Birr) is given by: $p(x) = 165 - 0.1x$, for $0 \leq x \leq 400$. The company is currently manufacturing 10 sets of computer per month, but is planning to increase production. What is the total change in the monthly profit if the monthly production increased to 60 sets? (1)
- A. Birr 50
B. Birr 1,865
C. ✓ Birr 8,075
D. Birr 18,635

Answer: C

The total change in profit is given by the integral of the marginal profit from 10 to 60: $\int_{10}^{60} (165 - 0.1x) dx = [165x - 0.05x^2]_{10}^{60} = (9900 - 180) - (1650 - 5) = 9720 - 1645 = 8075$ Birr.

49. If angle θ is an acute angle of a right triangle, what is the length of the side adjacent to θ , given the hypotenuse has 6 units length and $\sec \theta = 10/3$? (1)
- A. ✓ 1.8 units
B. 2 units
C. 18 units
D. 20 units

Answer: A

Since $\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$, we have $\text{adjacent} = \frac{\text{hypotenuse}}{\sec \theta} = \frac{6}{10/3} = \frac{18}{10} = 1.8$ units.

50. If $A = (1, -2)$, $B = (-3, 2)$ and $\vec{v}$ is a position vector such that $2\vec{v} + \overrightarrow{AB} = \vec{0}$, then $\vec{v}$ is equal to (1)
- A. (2, 0)
B. (-1, 0)
C. (-2, 2)
D. ✓ (2, -2)

Answer: D

The vector $\overrightarrow{AB} = B - A = (-3 - 1, 2 - (-2)) = (-4, 4)$. Since $2\vec{v} + \overrightarrow{AB} = \vec{0}$, $2\vec{v} = -(-4, 4) = (4, -4)$, which gives $\vec{v} = (2, -2)$.

51. If each of the compound propositions $P \vee Q$, $P \Rightarrow R$ and $\neg R$ is true, then which one of the following is True? (1)
- A. P
B. $\checkmark$ Q
C. $Q \Rightarrow P$
D. $P \wedge \neg R$

Answer: B

Since $\neg R$ is true, R is false. For $P \Rightarrow R$ to be true with R false, P must be false. Finally, since $P \vee Q$ is true and P is false, Q must be true.

52. If $\vec{A} = 4\vec{i} - 3\vec{j}$ and $\vec{u}$ is a unit vector such that $|\vec{A} + \vec{u}|^2 = 27$, then the cosine of the angle between $\vec{A}$ and $\vec{u}$ is equal to (1)
- A. $\checkmark$ 0.1
B. 0.2
C. 0.3
D. 0.4

Answer: A

We have $|\vec{A}| = \sqrt{4^2 + (-3)^2} = 5$ and $|\vec{u}| = 1$. Expanding gives $|\vec{A} + \vec{u}|^2 = |\vec{A}|^2 + 2\vec{A} \cdot \vec{u} + |\vec{u}|^2 = 25 + 2(5)(1) \cos \theta + 1 = 26 + 10 \cos \theta = 27$, so $10 \cos \theta = 1$, yielding $\cos \theta = 0.1$.

53. What is the possible value of x that solves the equation: $\sin^{-1} x + \cos^{-1} \left(\frac{3}{5}\right) = \pi$ (1)
- A. $\pi/3$
B. $3/5$
C. $5\pi/2$
D. $\checkmark$ $4/5$

Answer: D

54. If $P = (3, \alpha - 1, \alpha + 2)$ and $Q = (2\alpha + 1, 3, 3\alpha)$ are points in space, what should be the value(s) of α so that the distance between the two points is 6? (1)
- A. $\alpha = 1$, or $\alpha = 5$
B. $\checkmark$ $\alpha = \frac{4 + 2\sqrt{7}}{3}$, or $\alpha = \frac{4 - 2\sqrt{7}}{3}$
C. $\alpha = -1$, or $\alpha = 3$
D. $\alpha = -3$, or $\alpha = 2$

Answer: B

The distance squared between P and Q is $(2\alpha - 2)^2 + (4 - \alpha)^2 + (2\alpha - 2)^2 = 36$, which simplifies to $9\alpha^2 - 24\alpha - 12 = 0$, or $3\alpha^2 - 8\alpha - 4 = 0$. Solving via the quadratic formula gives $\alpha = \frac{8 \pm \sqrt{64 - 4(3)(-4)}}{6} = \frac{8 \pm 4\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$.

55. If $(-1, 2, 2)$ and $(1, 0, -2)$ are end points of a diameter of a sphere, then which one of the following is true about the sphere? (1)
- A. $(0, 1, 0)$ is a point on the sphere
B. $\checkmark$ the equation of the sphere is $x^2 + (y - 1)^2 + z^2 = 6$
C. the equation of the sphere is $x^2 + (y - 2)^2 + z^2 = 6$
D. the radius of the sphere is 6

Answer: B

The center of the sphere is the midpoint of the diameter: $\left(\frac{-1+1}{2}, \frac{2+0}{2}, \frac{2-2}{2}\right) = (0, 1, 0)$. The radius is the distance from the center to $(1, 0, -2)$, so $r^2 = (1 - 0)^2 + (0 - 1)^2 + (-2 - 0)^2 = 1 + 1 + 4 = 6$, making the equation $x^2 + (y - 1)^2 + z^2 = 6$.

56. Suppose that the equation $x^2 + y^2 + z^2 + 2x + 8z = 6(y + 1)$, represents a sphere. Where is the point (1, -1, 4) located relative to the sphere? (1)
- A. inside the sphere
B. on the sphere
C. at the center of the sphere
D. ✓ outside the sphere

Answer: D

Rewrite the equation as $x^2 + 2x + y^2 - 6y + z^2 + 8z = 6$, which completes squares to $(x + 1)^2 + (y - 3)^2 + (z + 4)^2 = 6 + 1 + 9 + 16 = 32$. Substituting the point (1, -1, 4) into the left-hand side gives $(1 + 1)^2 + (-1 - 3)^2 + (4 + 4)^2 = 4 + 16 + 64 = 84 > 32$, which means the point is outside the sphere.

57. Two ships, one with angle of depression 60° due east and the other with 30° due west are observed from a plane 1,000 metres above sea. If the two ships are on the same line, what is the distance between the two ships? [you may use the values: $\sin 30^\circ = \cos 60^\circ = 1/2$ and $\sin 60^\circ = \cos 30^\circ = \sqrt{3}/2$] (1)
- A. ✓ $\frac{1}{\sqrt{3}} 4,000 \text{ m}$
B. 2,000m
C. $\sqrt{3} 500 \text{ m}$
D. $\frac{1}{\sqrt{3}} 600 \text{ m}$

Answer: A

The horizontal distance to the east ship is $d_1 = \frac{1000}{\tan 60^\circ} = \frac{1000}{\sqrt{3}} \text{ m}$, and to the west ship is $d_2 = \frac{1000}{\tan 30^\circ} = 1000\sqrt{3} = \frac{3000}{\sqrt{3}} \text{ m}$. The total distance between the two ships is $d_1 + d_2 = \frac{4000}{\sqrt{3}} \text{ m}$.

58. What is the image of the ellipse $(x - 1)^2 + 4y^2 = 1$ under the translation that takes (1, 1) to (0, 2) followed, by the reflection through the x-axis? (1)
- A. $x^2 + 4(y - 1)^2 = 1$
B. $4x^2 + (y - 1)^2 = 1$
C. ✓ $x^2 + 4(y + 1)^2 = 1$
D. $4x^2 + (y + 1)^2 = 1$

Answer: C

The translation vector is $(0 - 1, 2 - 1) = (-1, 1)$, which transforms (x, y) to $(x - 1, y + 1)$, leading to the intermediate equation $x^2 + 4(y - 1)^2 = 1$. Reflecting through the x-axis replaces y with $-y$, yielding $x^2 + 4(-y - 1)^2 = x^2 + 4(y + 1)^2 = 1$.

59. What is the amplitude and period, respectively, of the graph of $f(x) = -6 \sin x \cos x$? (1)
- A. ✓ 3, π
B. 6, π
C. 3, $\pi/2$
D. 6, 2π

Answer: A

Using the double-angle identity, $f(x) = -6 \sin x \cos x = -3 \sin(2x)$. The amplitude is $|-3| = 3$, and the period is $2\pi/2 = \pi$.

60. What is the work done (in joule) when a force of 50 Newton is used to pull a cart 20 meters along path if the force is at an angle of 60° ? [Remember that: Work=(Force) x (distance travelled)] (1)
- A. 360
B. ✓ 500
C. 760
D. $1500\sqrt{2}$

Answer: B

Work is given by $W = F \cdot d \cos \theta = 50 \times 20 \times \cos(60^\circ) = 1000 \times 0.5 = 500 \text{ J}$.

61. Suppose L is the line through the center of the sphere $x^2 + y^2 + (z - 2)^2 = 9$ and the line intersects the sphere at $(1, 2, 4)$. What is the cosine of the angle between L and positive z -axis?
- A. $\checkmark$ $2/3$ B. $1/3$
C. $3/5$ D. $4/5$

Answer: A

The center of the sphere is $(0, 0, 2)$, giving a direction vector for line L of $\vec{v} = (1 - 0, 2 - 0, 4 - 2) = (1, 2, 2)$. The cosine of the angle with the positive z-axis $\vec{k} = (0, 0, 1)$ is $\frac{\vec{v} \cdot \vec{k}}{|\vec{v}|} = \frac{2}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}$.

62. Consider the formula for a natural number $n \in \mathbb{N}$: $2 + 4 + 8 + \dots + 2^n = 2^{n+1} + 1$ To proof this formula a person has used the following argument. "Assume the formula is true for $n=k$, for some $k \in \mathbb{N}$. Then the person has shown that the formula is also true for $n=k + 1$. And then, the person has concluded that, by the Principle of Mathematical induction, the formula is true for all natural numbers $n \in \mathbb{N}$." which one of the following statements is true about the above argument? (1)
- A. the formula holds true though it does not work for $n=1$
- B. since the left-hand-side is an even number and the right hand side an odd number, the Principle of Mathematical Induction is False.
- C. this is one example where the Principle of Mathematical Induction fails to work.
- D. ✓ the above formula does not work for all natural numbers $n \in \mathbb{N}$

Answer: D

Although the induction step holds algebraically, the base step $n = 1$ fails ($2 \neq 5$). In fact, the left side is always even and the right side is always odd, so the formula does not hold for any natural number.

— END OF MARKING KEY —